Exercise 1) Invariant Measure on SU(2)

$U = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$ with $a, b \in \mathbb{C}$ gives a parametrization of $SU(2)$. For $a = x + iy$ and $b = z + iw$ with $x, y, z, w \in \mathbb{R}$ we find that $x^2 + y^2 + z^2 + w^2 = 1$. Hence $SU(2)$ is homeomorphic to S^3.

Using polar coordinates

$$\begin{aligned} x &= r\cos\theta\cos\phi \\ y &= r\cos\theta\sin\phi \\ z &= r\sin\theta\cos\chi \\ w &= r\sin\theta\sin\chi \ , \end{aligned}$$

where $0 \leq \theta \leq \pi/2$, $0 \leq \phi$, $\chi \leq 2\pi$, we get for the Jacobian:

$$J = \begin{pmatrix} \cos\theta\cos\phi & -r\sin\theta\cos\phi & -r\cos\theta\sin\phi & 0 \\ \cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi & 0 \\ \sin\theta\cos\chi & r\cos\theta\cos\chi & 0 & -r\sin\theta\sin\chi \\ \sin\theta\sin\chi & r\cos\theta\sin\chi & 0 & r\sin\theta\cos\chi \end{pmatrix} .$$

This gives us

$$\begin{aligned} \det(J) &= \cos\theta\cos\phi(-r^3\cos^2\theta\sin\theta\cos^2\chi\cos\phi - r^3\cos^2\theta\sin\theta\sin^2\chi\cos\phi) \\ &\quad - \cos\theta\sin\phi(r^3\cos^2\theta\sin\theta\cos^2\chi\sin\phi + r^3\cos^2\theta\sin\theta\sin^2\chi\sin\phi) \\ &\quad + \sin\theta\cos\chi(-r^3\sin^2\theta\cos\theta\cos^2\phi\cos\chi - r^3\sin^2\theta\cos\theta\sin^2\phi\cos\chi) \\ &\quad - \sin\theta\sin\chi(r^3\sin^2\theta\cos\theta\cos^2\phi\sin\chi + r^3\sin^2\theta\cos\theta\sin^2\theta\sin\chi) \ , \end{aligned}$$

and hence

$$|\det(J)| = r^3\sin\theta\cos\theta \ .$$

For the normalization we calculate

$$\int_0^{2\pi}\int_0^{2\pi}\int_0^{\pi/2}\sin\theta\cos\theta d\theta d\phi d\chi = 4\pi^2\int_0^{\pi/2}\sin\theta\cos\theta d\theta = 4\pi^2(\frac{1}{2}\sin^2\theta|_0^{\pi/2}) = 2\pi^2 \ .$$

Thus we get $dg = \frac{1}{2\pi^2}\sin\theta\cos\theta$.

For

$$\begin{aligned} g &= \begin{pmatrix} \cos\theta\cos\phi + i\cos\theta\sin\phi & \sin\theta\cos\chi + i\sin\theta\sin\chi \\ -\sin\theta\cos\chi + i\sin\theta\sin\chi & \cos\theta\cos\phi - i\cos\theta\sin\phi \end{pmatrix} \\ &= \begin{pmatrix} \exp(i\phi)\cos\theta & \exp(i\chi)\sin\chi \\ -\exp(-i\chi)\sin\theta & \exp(-i\phi)\cos\theta \end{pmatrix} \end{aligned}$$

we get

$$g|0\rangle\langle 0|g^\dagger = \begin{pmatrix} \cos^2\theta & -\exp(i\phi + i\chi)\sin\theta\cos\theta \\ -\exp(-i\phi - i\chi)\sin\theta\cos\theta & \sin^2\theta \end{pmatrix} .$$

Hence the integral over the non-diagonal entries vanishes and we get

$$\begin{aligned} \int g|0\rangle\langle 0|g^\dagger dg &= \frac{1}{2\pi^2}\cdot 4\pi^2\int_0^{\pi/2}\begin{pmatrix} \cos^2\theta & 0 \\ 0 & \sin^2\theta \end{pmatrix}\sin\theta\cos\theta d\theta \\ &= \frac{1}{2}\begin{pmatrix} -\cos^4(\theta)/4|_0^{\pi/2} & 0 \\ 0 & \sin^4(\theta)/4|_0^{\pi/2} \end{pmatrix} = \frac{\mathbb{1}}{2} \ . \end{aligned}$$

Exercise 2) Schur's Lemma

Let $V_1 : SU(2) \to SU(2)$ with $V_1(h) = h$ for all $h \in SU(2)$ be the defining representation of $SU(2)$. This is an irreducible representation of $SU(2)$ on $\mathbb{C}^2$.

The term to determine, $\Phi := \int g|0\rangle\langle 0|g^\dagger dg$, is a homomorphism on $\mathbb{C}^2$. By the left-invariance of the Haar measure, we have for all $h \in SU(2)$

$$h\Phi h^\dagger = \int hg|0\rangle\langle 0|g^\dagger h^\dagger dg = \int hg|0\rangle\langle 0|(hg)^\dagger dg = \int g|0\rangle\langle 0|g^\dagger dg = \Phi \ .$$

Hence we have $h\Phi = \Phi h$. But by Schur's Lemma this implies $\Phi = \lambda \cdot \mathbb{1}$ for some $\lambda \in \mathbb{C}$.

Taking the trace

$$\mathrm{tr}(\int g|0\rangle\langle 0|g^\dagger dg) = \int \mathrm{tr}(g|0\rangle\langle 0|g^\dagger) dg = \int \mathrm{tr}(|0\rangle\langle 0|) dg = \int 1 dg = 1 \ ,$$

we get $\lambda = \frac{1}{2}$.