Exercise 1) Entanglement Dilution

a) As calculated in the script, the state after the Schur transform is given by

$$|\psi\rangle_{AB}^{\otimes n} = \sum_k c_k |\psi_k\rangle_{V_k^A \otimes V_k^A} \otimes |\phi_{m_k^n}\rangle_{[k]^A \otimes [k]^B} \ ,$$

where $|c_k|^2$ denotes the probability for k. Now the probability to measure P_ϵ can be calculated to

$$\begin{aligned} p_\epsilon &= \mathrm{tr}(|\psi\rangle\langle\psi|_{AB}^{\otimes n} P_\epsilon) = \mathrm{tr}(\rho_A^{\otimes n} P_\epsilon) = \mathrm{tr}(\rho_A^{\otimes n}(\sum_{k\in K} |k\rangle\langle k|)) = \sum_{k\in K} \mathrm{tr}(\rho_A^{\otimes n}|k\rangle\langle k|) \\ &= \sum_{k\in K} |c_k|^2 = 1 - \sum_{k\notin K} |c_k|^2 \ . \end{aligned}$$

where $K = n[1-2r-2\epsilon, 1-2r+2\epsilon]$. We continue with (cf. script page 24)

$$\begin{aligned} \sum_{k\notin K} |c_k|^2 &= \sum_{k\notin K} \binom{n}{\frac{n-k}{2}} r^{\frac{n-k}{2}} (1-r)^{\frac{n+k}{2}} \left(\frac{2k+2}{n+k+2} \cdot \frac{1-r}{1-2r}\right) \\ &\leq \frac{1-r}{1-2r} \cdot \sum_{k\notin K} \binom{n}{\frac{n-k}{2}} r^{\frac{n-k}{2}} (1-r)^{\frac{n+k}{2}} \\ &= \frac{1-r}{1-2r} \cdot \sum_{j\notin J} \binom{n}{j} r^j (1-r)^{n-j} \\ &= \frac{1-r}{1-2r} \cdot \left(1 - \sum_{j\in J} \binom{n}{j} r^j (1-r)^{n-j}\right) \ , \end{aligned}$$

where $j = \frac{n-k}{2}$ and $J = n[r+\epsilon, r-\epsilon]$. But by the law of large numbers we have

$$\lim_{n\to\infty} \left(\sum_{j\in J} \binom{n}{j} r^j (1-r)^{n-j}\right) = 1 \ ,$$

which let's us conclude that $p_\epsilon \to 1$ for $n \to \infty$.

(b) A twice-differentiable function $f(t)$ is concave if $f''(t) < 0$. For $f(t) = -t\log t$ we get $f''(t) = -\frac{1}{t}$, which is indeed smaller than zero for all $t > 0$.

Now define $g_s(t) = f(t+s) - f(t)$ for $s \in [0, \frac{1}{2}]$ and note that $g_s'(t) \leq 0$ for all $s \geq 0$. Hence we have for $t \in [0, 1-s]$ that

$$|g_s(t)| \leq \max\{g_s(0), g_s(1-s)\} \ ,$$

which is equivalent to

$$|f(t) - f(t+s)| \leq \max\{f(s), f(1-s)\} \ .$$

Furthermore we find that $f(1-s) \leq f(s)$ and hence $|f(t) - f(t+s)| \leq f(s)$.

Finally this gives us

$$\begin{aligned} |h(x) - h(x+\epsilon)| &\leq |f(x) - f(x+\epsilon)| + |f(1-x) - f(1-x-\epsilon)| && (1) \\ &\leq f(\epsilon) + f(\epsilon) = -2\epsilon\log\epsilon \ . && (2) \end{aligned}$$

The number of path ebits is given by $\log m_k^n$, and as shown on page 18 of the script we have

$$nh\left(\frac{1}{2}\left(1-\frac{k}{n}\right)\right) - 2\log(n+1) \leq \log m_k^n \leq nh\left(\frac{1}{2}\left(1-\frac{k}{n}\right)\right) \ .$$

By a) we know that $k \in n[1-2r-2\epsilon, 1-2r+2\epsilon]$, which gives us

$$nh(r+\epsilon) - 2\log(n+1) \leq \log m_k^n \leq nh(r+\epsilon)\ .$$

Using (2) we can conclude that

$$n(h(r)+2\epsilon\log\epsilon) - 2\log(n+1) \leq \log m_k^n \leq n(h(r)-2\epsilon\log\epsilon)\ .$$

c) The protocol needs entanglment to exchange the path ebits against ebits shared with Bob and to teleport all the remaining outputs from the Schur transform on the B systems to Bob.

For the first task we know from a) that between $n(h(r)+2\epsilon\log\epsilon) - 2\log(n+1)$ and $n(h(r) - 2\epsilon\log\epsilon)$ are needed. For the second task we need to teleport the remaining p registers, the l' register and the k register, for which we need

$$4n\epsilon\log\epsilon + 2\log(n+1) + 2\log n$$

ebits.

The classical communication needed comes from the teleportation step and hence we need

$$8n\epsilon\log\epsilon + 4\log(n+1) + 4\log n$$

bits of classical communication.

Exercise 2) Schmidt Coefficients

a) n ebits can be written as $|\psi\rangle_{AB}^{\otimes n} = (\frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB}))^{\otimes n}$ and the local density matrices become $\psi_A^n = \psi_B^n = \frac{1}{\sqrt{2^n}} \cdot \mathbb{1}_{2^n}$. The number of non-zero Schmidt coefficients is then equal to the rank of $\frac{1}{\sqrt{2^n}} \cdot \mathbb{1}_{2^n}$, which is given by 2^n.

b) Let $|\Psi'\rangle_{AB} = (P_A \otimes \mathbb{1}_B)|\Psi\rangle_{AB}$ be the (non-normalised) state after a local projection on Alice's side. Since the Schmidt coefficients are just the square roots of the eigenvalues of the local density matrix we find

$$\begin{aligned}\mathrm{rank}(\mathrm{tr}_B(|\Psi'\rangle\langle\Psi'|_{AB})) &= \mathrm{rank}(\mathrm{tr}_B((P_A\otimes\mathbb{1}_B)|\Psi\rangle\langle\Psi|_{AB}(P_A\otimes\mathbb{1}_B))) \\ &= \mathrm{rank}(P_A(\mathrm{tr}_B(|\Psi\rangle\langle\Psi|_{AB}))P_A)\ .\end{aligned}$$

Set $r := \mathrm{rank}(\mathrm{tr}_B(|\Psi\rangle\langle\Psi|_{AB}))$ and let $\mathrm{tr}_B(|\Psi\rangle\langle\Psi|_{AB}) = \sum_{i=1}^r \lambda_i |v_i\rangle\langle v_i|_A$ be an eigendecomposition. Then $P_A(\mathrm{tr}_B(|\Psi\rangle\langle\Psi|_{AB}))P_A = \sum_{i=1}^r \lambda_i |v_i'\rangle\langle v_i'|_A$ for $|v_i'\rangle_A = P_A|v_i\rangle_A$, and hence

$$\mathrm{rank}(P_A(\mathrm{tr}_B|\Psi\rangle\langle\Psi|_{AB})P_A) \leq \mathrm{rank}(\mathrm{tr}_B(|\Psi\rangle\langle\Psi|_{AB}))\ .$$