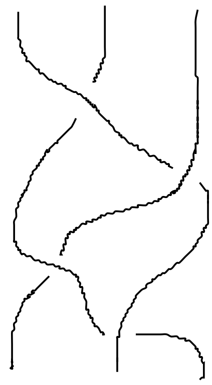
Exercise 1) Braid

The braid can represented by the following diagram



Exercise 2) Fibonacci Anyons

a) We first solve the pentagon equation to determine the F-matrix.

If one of the indices of the F-matrix is of trivial type, then it is easy to see that the matrix itself becomes trivial. That is

$$(F^0_{111})_{ab} = (F^1_{011})_{ab} = (F^1_{101})_{ab} = (F^1_{110})_{ab} = \delta_{a1} \cdot \delta_{b1} \ . \tag{1}$$

Checking all other possible indices (that are allowed by the fusion rules), we find that the only non-trivial F-matrix is $F^1_{111} \equiv F$. The pentagon equation can then be written as (notation as on page 41 in the script)

$$(F^1_{11k_{34}})_{k_{12}k_{234}}(F^1_{k_{12}11})_{k_{123}k_{34}} = (F^{k_{234}}_{111})_{k_{23}k_{34}}(F^1_{1k_{23}1})_{k_{123}k_{234}}(F^{k_{123}}_{111})_{k_{12}k_{23}} \ .$$

Setting $k_{34} = k_{123} = 0$ the pentagon equation simplifies to

$$(F^1_{110})_{k_{12}k_{234}}(F^1_{k_{12}11})_{00} = (F^{k_{234}}_{111})_{k_{23}0}(F^1_{1k_{23}1})_{0k_{234}}(F^0_{110})_{k_{12}k_{23}} \ ,$$

and together with (1) we obtain

$$(F^1_{111})_{00} = (F^1_{111})_{10}(F^1_{111})_{01} \ .$$

This combined with the condition that F^1_{111} is unitary constrains the matrix, up to arbitrary global phases, to be

$$F \equiv F^1_{111} = \begin{pmatrix} \varphi^{-1} & \varphi^{-1/2} \\ \varphi^{-1/2} & -\varphi^{-1} \end{pmatrix} \ , \tag{2}$$

where $\varphi = \frac{\sqrt{5}+1}{2}$ is the golden ratio.

As a second step we solve the hexagon equations to determine the R-matrix. There are two different hexagon equations. We solve the first one and find an up to global phase unique solution. It then turns out that the other hexagon equation is compatible with this solution.

Braiding with an anyon of trivial type is trivial, that is $R^0_{10} = R^1_{01} = 1$. Furthermore $F^1_{111} \equiv F$ is the only non-trivial F-matrix. Hence the first hexagon equation becomes (notation as on page 42 in the script)

$$R^c_{11}(F^1_{111})_{ac}R^a_{11} = \sum_{b=0,1}(F^1_{111})_{bc}R^1_{1b}(F^1_{111})_{ab} \ ,$$

for all $a, c \in \{0, 1\}$. With the help of (2) we then get

$$\begin{aligned}
&(R^0_{11})^2\varphi^{-1} = \varphi^{-2} + R^1_{11}\varphi^{-1} \\
&R^0_{11}R^1_{11}\varphi^{-1/2} = (1 - R^1_{11})\varphi^{-3/2} \\
&-(R^1_{11})\varphi^{-1} = R^1_{11}\varphi^{-2} + \varphi^{-1} \ ,
\end{aligned}$$

Solving this system of equations, we arrive at $R^0_{11} - = \exp(4\pi i/5)$ as well as $R^1_{11} = \exp(-3\pi i/5)$.

Thus we can conclude that

$$F = \begin{pmatrix} \frac{\sqrt{5}-1}{2} & \sqrt{\frac{\sqrt{5}-1}{2}} \\ \sqrt{\frac{\sqrt{5}-1}{2}} & \frac{1-\sqrt{5}}{2} \end{pmatrix}, \ R = \begin{pmatrix} e^{4\pi i/5} & 0 \\ 0 & e^{-3\pi i/5} \end{pmatrix} \ .$$

b) If the fusion of the first two particles yields trivial total charge, then the remaining $n-2$ particles can fuse in N^0_{n-2} distinguishable ways, and if the fusion of the first two particles yields an anyon with nontrivial charge, then that anyon can fuse with the other $n-2$ anyons in N^0_{n-1} ways. Therefore

$$N^0_n = N^0_{n-1} + N^0_{n-2} \ .$$

Since $N^0_1 = 0$ and $N^0_2 = 1$, the solution to this recursion relation is

$$\begin{array}{ccccccccc} n & = & 1 & 2 & 3 & 4 & 5 & 6 & \dots \\ N^0_n & = & 0 & 1 & 1 & 2 & 3 & 5 & \dots \end{array},$$

the dimensions are Fibonacci numbers. This can be rewritten as a matrix equality

$$\begin{pmatrix} N^0_{n+2} \\ N^0_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} N^0_{n+1} \\ N^0_n \end{pmatrix} \ ,$$

and hence

$$\begin{pmatrix} N^0_{n+1} \\ N^0_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} N^0_1 \\ N^0_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ .$$

The eigenvalues of $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ are $\frac{1+\sqrt{5}}{2}$, $\frac{1-\sqrt{5}}{2}$ and the corresponding eigenvectors are $\begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -\frac{1+\sqrt{5}}{2} \end{pmatrix}$.

Thus

$$\begin{pmatrix} N^0_{n+1} \\ N^0_n \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 1 \\ 1 & -\frac{1+\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix}^n \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 1 \\ 1 & -\frac{1+\sqrt{5}}{2} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ ,$$

from which we obtain

$$N^0_n = \frac{1}{\sqrt{5}} \cdot (\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}} \cdot (\frac{1-\sqrt{5}}{2})^n \ .$$

For large n we get $N^0_n = O(\varphi^n)$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.