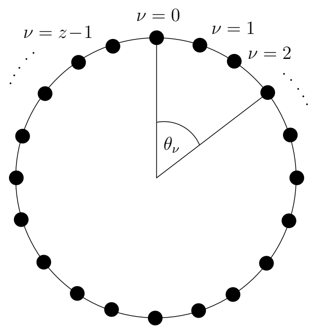
Statistical Physics
Exercise Sheet 2

Exercise 2.1 Statistical Polarization

We study a system of N compasses with a needle pointing in one of z directions. The needle hops to a neighboring direction with a rate Γ (nearest neighbor interaction). We assume that z is even for ease of presentation.



a) Express the master equation and the entropy (H-function) of this system in terms of the numbers N_ν of compasses pointing in direction ν, where $\nu \in \{0, 1, \dots, z-1\}$. What is the equilibrium state of this system?

b) The system is prepared with all the needles pointing uniformly distributed to the positions $\nu \in \{0, 1, \dots, z/2 - 1\}$. What is the entropy of the initial distribution? Compare your result with the equilibrium entropy and interpret the result with respect to the ideal gas entropy.

c) At time $t = 0$, all needles point in the same direction ($N_0 = N$, $N_\nu = 0 \ \forall \nu \neq 0$). Calculate for long times ($t \gg z^2/\Gamma$) the polarization of the system

$$P(t) := \langle \cos\theta \rangle(t) = \frac{1}{N} \sum_\nu N_\nu(t) \cos\theta_\nu \,, \quad \theta_\nu = \frac{2\pi\nu}{z} \,, \tag{1}$$

and compare the relaxation of the polarization with the one for the entropy.
Hint: Use the Fourier transform

$$N_\nu(t) = \sum_{k=-z/2}^{z/2-1} \tilde{N}_k(t)\, e^{-i\frac{2\pi}{z}k\nu}$$

and express the master equation and $P(t)$ in terms of the coefficients $\tilde{N}_k(t)$.

d) Starting with the same initial distribution as in c), calculate the exact time dependence of N_ν for the case of $z \to \infty$.
Hint: The N_ν can be written in terms of Bessel-functions using the Jacobi-Anger expansion

$$e^{ix\cos\theta} = \sum_{n=-\infty}^{\infty} i^n J_n(x) e^{in\theta} \,. \tag{2}$$

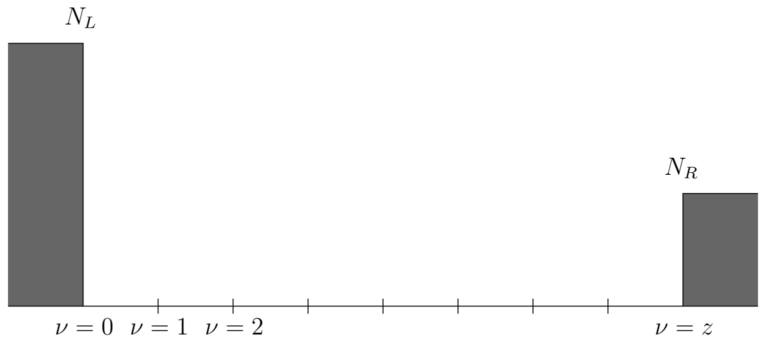
e) We go to a continuum description: $z \to \infty$ with $L = za$ constant, where a is the distance between points. We find the diffusion equation

$$\dot{\rho}(x,t) = D\partial_x^2 \rho(x,t)\,. \tag{3}$$

What is D in this equation?
Starting with the same initial distribution as in c), calculate the time dependence of the entropy for $L \to \infty$ (i.e. ignoring boundary effects). Interpret your result with respect to the ideal gas.

Exercise 2.2 Particle Current and Entropy Production

We consider a chain with z sites that is connected to two reservoirs such that the boundary conditions $N_0 = N_L$ and $N_z = N_R$ hold for all times (see figure).



a) Find the stationary solution and show that the entropy is not a constant in this case. What process is responsible for the increase in entropy?

b) We can define a particle current J_ν on the bond between the sites ν and $\nu + 1$ through the continuity equation $\dot{N}_\nu = J_{\nu+1} - J_\nu$. Consider the stationary state close to the equilibrium, where

$$N_L - N_R = \epsilon\,, \qquad 0 < \epsilon \ll N_L\,.$$

Show that while the current is proportional to ϵ, the entropy production is proportional to ϵ^2. What does that mean for the description of transport phenomenon (heat, particles) close to equilibrium?