

Exercise 13.1 Dielectric Susceptibility of Free Electrons

Consider a non-interacting one-dimensional gas of spinless electrons,

$$\mathcal{H} = \frac{1}{L} \sum_k \frac{k^2}{2m} c_k^\dagger c_k \quad \text{with } \hbar = 1. \quad (1)$$

We want to evaluate its linear response to an external scalar potential, i.e. a perturbation of the form

$$\delta\mathcal{H} = -e \int dx \phi(x, t) \hat{n}(x) = -e \int dx \phi(x, t) c^\dagger(x) c(x). \quad (2)$$

The density operator $\hat{n}(x)$ is defined by the second equality.

- a) To compute the linear response, use the Kubo-formalism (section 6.1) for the dielectric susceptibility,

$$\chi_e(x - x', t - t') = -i\Theta(t - t') e^2 \langle [\hat{n}_H(x, t), \hat{n}_H(x', t')] \rangle_{\mathcal{H}}, \quad (3)$$

where $\hat{n}_H(x, t)$ represents the density operator in the Heisenberg picture. Show that the Fourier transform $\chi(q, \omega)$ of $\chi_e(x - x', t - t')$ can be written as

$$\chi(q, \omega) = e^2 \sum_k \frac{f(\epsilon_{k+q}) - f(\epsilon_k)}{\omega + \epsilon_k - \epsilon_{k+q} + i\eta}, \quad (4)$$

where $f(\epsilon)$ denotes the Fermi function.

- b) The imaginary part of the so-called Lindhard function $\chi(q, \omega)$ obtained in a) encodes the spectrum of the (charge) excitations that couple to $\phi(x, t)$. Derive conditions for the region in the (q, ω) -plane for which $\text{Im } \chi(q, \omega) \neq 0$ holds and make a schematic graph. Argue that the “particle-hole excitations” also fulfill the same conditions.

Hint: In order to consider excitations from the ground state, consider the zero temperature limit and first show that $\chi(q, \omega)$ can be written as

$$\chi(q, \omega) = \sum_{|k| \leq k_F} \left[\frac{1}{\omega + \epsilon_k - \epsilon_{k+q} + i\eta} - \frac{1}{\omega - \epsilon_k + \epsilon_{k+q} + i\eta} \right]. \quad (5)$$

Then, take the continuum limit and obtain $\text{Im } \chi(q, \omega)$ using the Dirac identity

$$\frac{1}{x \pm i0} = \mathcal{P} \frac{1}{x} \mp i\pi\delta(x), \quad (6)$$

where \mathcal{P} denotes the Cauchy principal value, and integration over x is implied.

Office Hours: Monday, 20th of December, 10–12 am (HIT K 31.3)