1. Effective Action

In a quantized field theory, we compute expectation values and physical observables through a "partition functional."

$$Z[J] = e^{iW[J]} =$$

$$= \int_{\substack{\text{all} \\ \text{field paths}}} \mathcal{D}\varphi \; e^{i\left(S[\varphi] + \int d^4x \, J(\varphi)\,\varphi(x)\right)}$$

For simplicity we consider here one only field (scalar) and one source term $J(x)$ but this discussion can be generalized.

($Z[J]$ is the sum of all Feynman diagrams, while $W[J]$ is the sum of all connected Feynman diagrams.).

Expectation values of field operators in the ground state can be obtained by acting with functional derivatives $\left(\frac{1}{i}\frac{\delta}{\delta J(x_i)}\right)$ on the $W[J]$. In the simplest cases.

$$\langle \text{VACUUM} | \text{VACUUM} \rangle_J = Z[J] \qquad (\text{Eq. 1})$$

$$\langle \text{VAC} | \hat{\phi}(x_1) | \text{VAC} \rangle = \frac{\delta}{\delta J(x_1)} Z[J] \qquad (\text{Eq. 2})$$

and so on.

The vacuum expectation value of the field operator is: ~~defined via~~:

$$\langle \hat{\phi}(x) \rangle_J \equiv \frac{\langle VAC | \hat{\phi}(x) | VAC \rangle_J}{\langle VAC | VAC \rangle_J} =$$

$$= \frac{1}{i} \frac{1}{Z[J]} \frac{\delta}{\delta J(x)} Z[J] =$$

$$\Rightarrow \boxed{\langle \hat{\phi}(x) \rangle_J = \frac{\delta}{\delta J(x)} W[J]} \quad (\text{Eq. } 3)$$

We have considered a situation where the system is "fed" with an external source $J(x)$. The more interesting situation with $J(x) = 0$ gives the field vacuum expectation value of a closed system which is left to relax on its ground state.

We can integrate the above differential equation, and obtain:

$$W[J] = \int d^4x \, \langle \phi(x) \rangle_J \, J(x) + \underbrace{\Gamma[\langle \phi \rangle_J]}_{\text{"Constant of integration"}} \quad (\text{Eq. } 4)$$

$\Gamma[\langle\phi\rangle_J]$ depends on $\langle\phi\rangle_J$ (and only through it depends on the source J).

Differentiating (Eq. 4) with the field vev, we obtain

$$\frac{\delta\,\Gamma[\langle\phi\rangle_J]}{\delta\langle\phi\rangle_J(x)} = \frac{\delta W[J]}{\delta\langle\phi\rangle_J(x)} - \frac{\delta}{\delta\langle\phi\rangle_J(x)} \int d^4y\; J(y)\langle\phi\rangle_J(y)$$

$$= \int d^4y\; \frac{\delta J(y)}{\delta\langle\phi\rangle_J(x)} \underbrace{\frac{\delta W[J]}{\delta J(y)}}_{\parallel} - J(x)$$

$$- \int d^4y\; \frac{\delta J(y)}{\delta\langle\phi\rangle_J(x)} \langle\phi\rangle_J(y)$$

$$\rightsquigarrow \quad \boxed{\frac{\delta\,\Gamma[\langle\phi\rangle_J]}{\delta\langle\phi\rangle_J(x)} = -J(x)} \quad \text{(Eq. 5)}$$

Consider the physical case of interest with no external sources (closed system/universe)

Then $$\frac{\delta\Gamma[\langle\phi\rangle]}{\delta\langle\phi\rangle} = 0$$

and the vacuum expectation value of the

field operator corresponds to a stationary point of the effective action.

2. Symmetries of the effective action

Assume that the action $S[\varphi]$ is symmetric unde

$$\varphi \longleftrightarrow -\varphi . \; : \; S[\varphi] = S[-\varphi]$$

Then the effective action is sharing the same symmetry.

$$\Gamma[\langle\varphi\rangle_J] = \Gamma[-\langle\varphi\rangle_J]. \quad \text{(Eq. 6)}$$

Proof:

The functional

$$e^{+iW[J]} = \int D\varphi \; e^{i[S[\varphi] + \int d^4x \, J(x)\, \varphi(x)} =$$

$$= \int \cancel{D(\varphi)} \cdot e^{i(S[-\varphi] + \int d^4x \, (-J(x)) \cdot (-\varphi(x)))}$$

$$\overset{\bar\varphi = -\varphi}{=} \int D\bar\varphi \; e^{i(S[\bar\varphi] + \int d^4x \, (-J(x))\, \bar\varphi(x))}$$

$$= e^{+iW[-J]} \quad \leadsto$$

$$\leadsto \boxed{W[-J] = W[J]} \qquad \text{(Eq. 7)}$$

The vacuum expectation value of the presence of a source $J_p(x) = -J(x)$ is

$$\langle \phi \rangle_{J_p}^{(x)} = \frac{\delta \Gamma[J_p]}{\delta J_p(x)} \Rightarrow$$

$$\Rightarrow \langle \phi \rangle_{-J}^{(x)} = \frac{\delta W[-J]}{\delta(-J(x))} = -\frac{\delta W[-J]}{\delta J(x)} =$$

$$\overset{(Eq.7)}{=} -\frac{\delta W[J]}{\delta J(x)} \rightsquigarrow$$

$$\rightsquigarrow \langle \phi \rangle_{-J}^{(x)} = -\langle \phi \rangle_J(x) \qquad (Eq. 8)$$

Then we can easily arrive to

$$\Gamma[-\phi_J(x)] = \Gamma[\phi_{-J}(x)] =$$

$$= W[-J] - \int d^4x \langle \phi \rangle_{-J}(x) \, (-J(x)) =$$

$$= W[J] - \int d^4x \left(-\langle \phi \rangle_J(x)\right)\left(-J(x)\right) =$$

$$= W[J] - \int d^4x \, \langle \phi \rangle_J(x) \, J(x)$$

$$= \Gamma[\phi_J(x)].$$

Does the effective action share always the symmetries of the classical action? No! Let's see when exactly this happens.

Take a symmetry transformation

$$\varphi \to \varphi + \epsilon \underbrace{F(x,\varphi)}_{\text{functional of } \varphi.}$$

which leaves $D(\varphi + \epsilon F(x,\varphi)) = D\varphi$

and $S[\varphi + \epsilon F(x,\varphi)] = S[\varphi]$.

The partition functional under this transformation becomes,

$$Z[J] = \int D(\varphi + \epsilon F)\; e^{i\left[S[\varphi+\epsilon F] + \int d^4y\,(\varphi+\epsilon F)J\right]}$$

$$= \int D(\varphi)\; e^{i\left\{\left(S[\varphi] + d^4y\, \varphi(y) J(y)\right) + \epsilon \int d^4y\, F(y) J(y)\right\}}$$

which after Taylor expansion in ϵ gives:

$$Z[J] = Z[J] + \epsilon \cdot Z[J] \int d^4y\, \langle F(y, \varphi(y)) \rangle_J\, J(y)$$

with

$$\langle F(y,\varphi(y)) \rangle_J = \frac{\int D\varphi\; e^{i[S[\varphi] + \int d^4y\, J\cdot\varphi]} \cdot F(y,\varphi(y))}{Z[J]}$$

the "quantum average" of the functional $F(y,\varphi(y))$, The transformation $\varphi \to \varphi + \epsilon F(x, \varphi)$ is also ~~assumed~~ to be a symmetry transformation of $Z[J]$ at the quantum level. This only happens if

$$\varphi \to \varphi + \epsilon F \quad : \quad Z[J] \to Z[J]$$

if

$$0 = \int d^4y \, \langle F \rangle_J^{(y)} \, J(y)$$

But

$$J(y) = - \frac{\delta \Gamma[\langle \phi \rangle_J]}{\delta \langle \phi \rangle_J (y)}$$

$\leadsto$

$$\boxed{\int d^4y \, \langle F \rangle_J (y) \, \frac{\delta \Gamma[\langle \varphi \rangle_J]}{\delta \langle \phi \rangle_J (y)} = 0}$$

(Slavnov-Taylor identities).

We have just proven that if

$\varphi \to \varphi + \epsilon F$ is a symmetry transformation of the partition function $Z[J]$, then

$\langle \varphi \rangle \to \langle \varphi \rangle + \epsilon \langle F \rangle_J$ is a symmetry transformation of the effective action $\Gamma[\varphi]$. The two transformations are not necessarily the same; since

F may or may <u>not</u> be $\langle F \rangle$!

For <u>linear transformations</u> it is

$$F = \langle F \rangle . \quad \text{[Exercise.]}$$

Indeed, consider

$$\phi_i \to \phi_i'(x) = \phi_i(x) + \underbrace{\epsilon \int d^4y\, T_{ij}(x,y)\, \phi_j(y) + S_i(x)}_{F}$$

and

$$\langle F \rangle = S(x) + \epsilon \int T_{ij}(x,y) \langle \varphi_j \rangle_{J_\varphi}(y)$$

But

$$\langle \varphi \rangle_{J_\varphi} = \varphi . \quad \text{(Prove).}$$

So $\langle F \rangle = F$.