WEEK 2

Recap. & some further points on last lecture.

* For a quantized field theory with a generating functional

$$Z[J] = e^{iW[J]}$$

we can compute the quantum effective action

$$\Gamma[\langle\phi\rangle] = W[J] - \int d^4x \, J(x) \langle\phi\rangle(x).$$

The vacuum expectation value in the presence of a source J is:

$$\langle\phi\rangle_J = \frac{\langle 0|\phi(x)|0\rangle_J}{\langle 0|0\rangle_J} = \frac{\delta W[J]}{\delta J(x)}$$

Also,

$$\frac{\delta\Gamma[\langle\phi\rangle]}{\delta\langle\phi\rangle_J(x)} = -J(x)$$

The action $\Gamma[\langle\phi\rangle]$ has an extremum for a physical $(J=0)$ vacuum expectation value of the field operator.

$\rightarrow$ for $J=0 \rightsquigarrow \dfrac{\delta\Gamma}{\delta\langle\phi\rangle} = 0$!

* Assume that $Z[J]\big|_{J=0}$ is symmetric

under a symmetry transformation

$$\phi_i \to \phi_i' = \phi_i + \delta\phi_i =$$

$$= \phi_i + F[\varphi_i]$$

Then the effective action is symmetric under

$$\langle\phi_i\rangle \to \langle\varphi_i\rangle + \langle F[\varphi_i]\rangle$$

For discrete symmetries or linear symmetry transformations ~~which~~ we have

$$\langle F[\varphi_i]\rangle = F[\langle\varphi_i\rangle]$$

and the symmetries of the action are also symmetries of the effective action.

Spontaneous symmetry breaking

Symmetric transformations which ~~leave~~ the ~~quantum system~~ effective action intact may not be symmetries of the physical states and the vacuum ~~state~~. These symmetries are "spontaneously broken".

Spontaneous symmetry breaking is associated with a degeneracy of ~~the~~ ground state (vacuum). Assume that an effective action is symmetric under

$$\langle \phi \rangle \rightarrow - \langle \phi \rangle ,$$

a symmetry which is inherited from the classical action (as we have seen last week). Assume ~~also~~ that the physical field vev is

$$\langle \phi \rangle_a : \frac{\delta \Gamma}{\delta \langle \phi \rangle_\alpha} = 0 \quad \text{with}$$

$\langle \phi \rangle_a \neq 0$. ~~There~~ In other words, there is a vacuum state V_a:

$$\langle \phi \rangle_a = \frac{\langle V_\alpha | \hat{\phi}(x) | V_\alpha \rangle}{\langle V_a | V_a \rangle} \neq 0 .$$

for which $\Gamma[\langle \phi \rangle_a] = \Gamma_a$ is minimum.

Then, there should be a second value of the field vev in a different state which also ~~has~~ gives the same value for the effective action:

$$\exists \; |\beta\rangle : \langle \phi \rangle_\beta = - \langle \phi \rangle_\alpha :$$

$$\Gamma[\langle \phi \rangle_\beta] = \Gamma[\langle \phi \rangle_a] = \text{minimum} .$$

So, while $\phi \to -\phi$ preserves

$$\Gamma[\langle\phi\rangle] \longrightarrow \Gamma[-\langle\phi\rangle] = \Gamma[\langle\phi\rangle],$$

under the same transformation

$$|V_\alpha\rangle \longrightarrow |V_\beta\rangle$$ which is

a different state. The symmetry is broken, as long as the true vacuum is one of the two states.

But these are not the only possibilities. How do we know that the system ~~Could the system live on a non diagonal~~ will choose one of the two states to live in, and do not tunnel from one to another?

By symmetry we know that

$$\langle V_\alpha|H|V_\alpha\rangle \overset{(\phi\to-\phi)}{=} \langle V_\beta|H|V_\beta\rangle = E_{diag}$$

$$\text{and } \langle V_\alpha|H|V_\beta\rangle = \langle V_\beta|H|V_\alpha\rangle = E_{non\text{-}diag}.$$

We can compute that

$$E_{non\text{-}diag} = 0$$

in field theory, where volume is taken to be infinite, for any Hamiltonian.

Causality ~~Hermitian~~ $\to \vec{x}$ outside light-cone.

$$\langle u | \underbrace{[B(0), A(\vec{x})]}_{=0} | v\rangle = \langle u | 0 | v\rangle$$

$$\sim \sum_w \langle u | A(0) | w\rangle \langle w | B(0) | v\rangle$$

$$= \sum_w \underline{\langle u | B(0) | w\rangle} \underline{\langle w | A(0) | v\rangle}.$$

Take ~~$A(x)$~~ $\hat{A}, \hat{B}$ Hermitian.

Then, $\sum_w [\langle u | A(0) | w\rangle, \langle u | B(0) | w\rangle] = 0$

for any $\hat{A}$ & B. All $\langle u | A(0) | v\rangle$ can be simultaneously diagonalized. and so

$$\langle u | A(0) | v\rangle = \underline{\underline{\delta_{uv}}}\, a_v$$

$a_v \in$ Real.

For $u \neq v \sim$

$$\langle u | A(0) | v\rangle = 0$$

($\langle u|v\rangle = \delta_{uv}$)

Different ~~vac~~ but degenerate vacua, diagonalize any perturbation made out of local operators which break the symmetry. Linear combinations do not diagonalize such perturbations → and are therefore unstable.

Goldstone Bosons !

Take a symmetry transform which is linear,

$$\phi_n(x) \to \phi_n(x) + i\epsilon \sum_m t_{nm}\, \phi_m(x)$$

and leaves the ~~effective~~ action, and thus also the effective action, invariant.

$$\Gamma[\langle\phi_n\rangle + i\epsilon \sum_m t_{nm} \langle\phi_m\rangle] = \Gamma[\langle\phi_n\rangle] \rightsquigarrow$$

(Slavnov-Taylor)

$$\rightsquigarrow \int d^4x \, \frac{\delta \Gamma[\langle\phi\rangle]}{\delta \phi_n(x)} \, t_{nm} \phi_m(x) = 0.$$

Taking a second derivative,

$$\int d^4x \, \frac{\delta^2 \Gamma[\phi]}{\delta \phi_\ell(y)\, \delta\phi_n(x)} \, t_{nm} \langle \phi_m(x)\rangle +$$

$$+ \frac{\delta \Gamma[\phi]}{\delta \phi_\ell(y)} = 0.$$

Consider physical systems where $(J=0)$ and

$$\frac{\delta \Gamma}{\delta \phi}\bigg|_{\phi=\langle\phi\rangle} = 0 .$$

$$\rightsquigarrow \int d^4x \, \frac{\delta^2 \Gamma[\phi]}{\delta\langle\phi_\ell(y)\rangle \, \delta\langle\phi_m(x)\rangle} \, t_{nm} \langle \phi_m(x)\rangle = 0$$

For systems with translational invariance,

$$\langle \phi(x) \rangle = \int \langle 0 | \, e^{i\hat{P}\cdot x} \, \phi(0) \, e^{-i\hat{P}\cdot x} | 0 \rangle \, d^4 p$$

$$= \delta(0) \, \langle \phi \rangle_0 = \text{constant} .$$

$$\langle \phi_\ell(y) \rangle = \langle \phi_\ell \rangle$$

Also $\Gamma[\langle\phi\rangle] = -\int d^4x \; V(\langle\phi\rangle)$

and, we get.

$$\boxed{\sum_{m,n} \frac{\partial^2 V(\phi)}{\partial\langle\phi_\ell\rangle \, \partial\langle\phi_m\rangle} \, t_{nm} \langle \phi_m \rangle = 0} \qquad (\text{Eq. } 1)$$

Let's take.

$$\frac{\delta W[J]}{\delta J_m(x)} = \langle \phi_m(x) \rangle$$

$$\rightsquigarrow \frac{\delta^2 W[J]}{\delta \langle \phi_n(y) \rangle \, \delta J_m(x)} = \delta(x-y) \, \delta_{nm}$$

$$\rightsquigarrow -\delta(x-y) \, \delta_{nm} = \int d^4z \, \frac{\delta^2 W[J]}{\delta J_k(z) \, \delta J_m(x)}$$

$$\cdot \left(- \frac{\delta J_k(z)}{\delta \langle \phi_n \rangle (y)} \right) =$$

$$= \int d^4z \, \langle 0 | T \phi_k(z) \phi_n(x) | 0 \rangle \cdot$$

$$\cdot \frac{\delta^2 \Gamma}{\delta \langle \phi_n \rangle (y) \, \delta \langle \phi_k \rangle (z)}$$

But, $\langle 0 | T \phi_k(z) \, \phi_n(x) | 0 \rangle =$

$$= \int d^4p \, e^{-ip(x-z)} \, D_{kn}(p)$$

Inserting above,

$$\overset{y\to 0}{\rightsquigarrow} \quad -\overset{\delta_{nl}}{\delta}(x) = -\delta(x) \int d^4z \, d^4p \, e^{-ip(x-z)}$$

$$D_{kl}(p) \frac{\partial^2 V(\epsilon_e)}{\partial\langle\phi_n\rangle \partial\langle\phi_k\rangle}$$

$$\rightsquigarrow \boxed{\delta_{nl} = \sum_k \frac{\partial^2 V}{\partial\langle\phi_n\rangle\partial\langle\phi_k\rangle} D_{kl}(0)}$$

$$\rightsquigarrow \boxed{\frac{\partial^2 V}{\partial\langle\phi_n\rangle\partial\langle\phi_l\rangle} = D^{-1}_{nl}(0)}$$

Going back,

$$\rightsquigarrow \boxed{\sum_{n} D^{-1}_{ln}(0)\, t_{nm} \langle\phi_m\rangle = 0} \quad (\text{Eq. 2})$$

When is Eq. 2 satisfied? Let us write,

$$\delta\langle\phi_n\rangle = i\epsilon \sum_m t_{nm} \langle\phi_m\rangle$$

~~if $\delta\langle\delta$~~

Then.

$$\text{Eq. 2} \rightsquigarrow \boxed{\sum_n D^{-1}_{ln}(0) \left(\delta\langle\phi_n\rangle\right) = 0}$$

(Eq. 3).

If symmetry leaves the vacuum state unchanged, $\delta\langle\phi_n\rangle = 0$, then (Eq-?) is satisfied. ~~For generators with $\delta\langle\phi_n\rangle \neq 0$.~~
If the symmetry is broken

$$\left(\Delta^{-1}_{\ell m}(0)\right) \underbrace{\left(\langle\phi_m\rangle\right)} = 0.$$

eigen-vector of

with zero eigenvalue..

So. $\Delta^{-1}_{\ell m}(p) \to 0$ as $p \to 0$ and

$\Delta_{\ell n}(p) \sim \frac{1}{p^2}$ has a pole as

$p^2 = 0$. $\Longrightarrow$ Massless particle in the spectrum of the theory. $\Rightarrow$ "Goldstone Bosons".

<u>Second proof:</u>

Symmetry $\sim$ conserved current

$J^{\mu}(x)$