Spontaneously Broken Approximate Symmetries

Consider a potential

$$V_{eff}(\varphi) = V_0(\varphi) + V_I(\varphi) \qquad (\text{Eq. } 1)$$

where

$V_0(\varphi)$ is invariant under some symmetry transformation

$$\langle \varphi_n \rangle \to \langle \varphi_n \rangle + \langle \delta \varphi_n \rangle \cdot \epsilon$$

with $\delta \varphi_n \equiv T^\alpha_{nm} \phi_m$ (where T^α_{nm} generators of the symmetry group).

Requiring

$$\delta V_0(\langle \varphi \rangle) = 0 \sim$$

$$\sim \sum_{nm} T^\alpha_{nm} \langle \phi_m \rangle \frac{\partial V_0}{\partial \langle \phi_n \rangle} = 0 \qquad (\text{Eq. } 2).$$

We will take $V_I(\varphi)$ to not respect the same symmetry and break it explicitly,

$$\delta V_I(\varphi) \neq 0 \quad \text{if} \quad \varphi \to \varphi + \delta \varphi . \quad (\text{Eq. } 3)$$

Let's assume that $V_I(\varphi)$ is relatively small.

Then, the symmetry, although not exact, is considered to hold approximately.
We have studied the spontaneous breaking of exact symmetries earlier. Is it also occurring to an approximate symmetry? In other words, should we expect that the ground state varies by applying a symmetry transformation? This is indeed the case. In contrast to SSB of an exact global symmetry, we will find however that the vacuum state which is realized is constrained by the symmetry breaking term $V_I(\varphi)$ in the potential.

The vacuum state is found by solving

$$\frac{\partial V}{\partial\langle\varphi_n\rangle} = 0 .$$

Let us expand around the vacuum of the system with an exact symmetry ($V_I(\varphi) = 0$),

$$\langle\phi_n\rangle = \langle\phi_n^0\rangle + \langle\phi_n^I\rangle$$

with $$\frac{\partial V_0}{\partial\langle\phi_n^0\rangle} = 0 .$$

$$\Rightarrow \sum_m \left(\sum_n \frac{\partial^2 V_0}{\partial \langle \phi_m^0 \rangle \partial \langle \phi_n^0 \rangle} \delta \langle \phi_n \rangle \right) \langle \phi_m^I \rangle$$

$$+ \sum_n \frac{\partial V_I}{\partial \langle \phi_n^0 \rangle} \delta \langle \phi_n^0 \rangle = 0$$

The term in $(\cdots)$ is zero, if we recall the proof of Goldstone's theorem based on the effective action formalism. We therefore have:

$$\boxed{\sum_{nm} \frac{\partial V_I}{\partial \langle \phi_n^0 \rangle} T_{nm} \langle \phi_m^0 \rangle = 0}$$

This is the "vacuum alignment condition". We can prove that $\exists \langle \phi_m^0 \rangle$ which satisfies it.

The vacuum state $|0\rangle$ is forced into an alignment to the symmetry breaking term V_I.

For spontaneously broken exact global symmetries we found that there must exist massless Goldstone boson scalars.

What is the mass of such particles when the symmetry is approximate?

Proof follows later.

Let us recall some of the properties of Goldstone boson states

$$|\beta a\rangle$$

in a theory with a spontaneously broken exact symmetry group. These states can be created by field operators acting on the vacuum:

$$\langle \beta_a | \phi_n(x) | 0 \rangle = \frac{e^{+ip_\beta \cdot x} \, Z_{an}}{(\sqrt{2\pi})^3 \sqrt{2 p_\beta^0}}$$

and have the quantum numbers of the conserved currents:

$$\langle 0 | J_a^\alpha(x) | \beta_b \rangle = \frac{i \, F_{ab} \, p_\beta^\alpha \, e^{-ip_\beta \cdot x}}{(2\pi)^{3/2} \sqrt{2 p_\beta^0}}.$$

We have proven that:

$$i \sum_b F_{ab} Z_{bn} = - \sum_m T^\alpha_{nm} \langle \Phi_m \rangle.$$

$$(\mathbf{F} \cdot \mathbf{Z} = \mathbf{v})$$

Let us now rewrite the Lagrangian system in terms of fields π^α which create exactly the Goldstone boson states:

$$\langle \beta_\alpha | \Pi_\beta | 0 \rangle = \frac{e^{+ip_\beta \cdot x} \, \delta_{a\beta}}{(2\pi)^{3/2} \sqrt{2p_\beta^0}}$$

We can rewrite:

$$\phi_n(x) = \sum_\alpha C_{n\alpha} \Pi_a(x) + \text{other fields not creating Goldstone bosons}$$

$$\rightsquigarrow \langle \beta_a | \phi_n(x) | 0 \rangle = \sum_\beta C_{n\beta} \langle \beta_a | \Pi_\beta(x) | 0 \rangle + \cancel{\ }$$

$$\Rightarrow Z_{an} = \sum_\beta C_{n\beta} \, \delta_{a\beta} \Rightarrow$$

$$\Rightarrow \boxed{C_{na} = Z_{an}}$$

We therefore obtain

$$\langle \phi_n \rangle \equiv \langle 0 | \phi_n(x) | 0 \rangle = \sum_\alpha Z_{an} \langle 0 | \Pi_a(\cancel{x}) | 0 \rangle$$

vev's of other fields

$$\rightsquigarrow \boxed{\langle \phi_n \rangle = \sum_\alpha Z_{an} \langle \Pi_a \rangle} + \dots$$

To compute the mass matrix of the physical Goldstone bosons, we have:

$$M^2_{a\beta} = \frac{\partial^2 V_{eff}}{\partial \langle \Pi_\alpha \rangle \partial \langle \Pi_\beta \rangle} =$$

$$= \sum_{nm} \frac{\partial \langle \phi_n \rangle}{\partial \langle \Pi_a \rangle} \frac{\partial \langle \phi_m \rangle}{\partial \langle \Pi_\beta \rangle} \frac{\partial^2 V_{eff}}{\partial \langle \phi_n \rangle \partial \langle \phi_m \rangle}$$

$$\rightsquigarrow M^2_{\alpha\beta} = \sum_{nm} Z_{an} Z_{\beta m} \frac{\partial^2 V_{eff}}{\partial \langle \phi_n \rangle \cdot \partial \langle \phi_m \rangle}$$

For an exact symmetry, which is broken spontaneously, <u>all</u> Goldstone bosons are massless and thus

$$M^2_{\alpha\beta} = 0 \qquad (\text{for exact symmetries}).$$

In the case of an approximate symmetry we write:

$$M^2_{ab} = \sum_{mn} Z_{an} Z_{bm} \left. \frac{\partial^2 V_{eff}}{\partial \phi_n \partial \phi_m} \right|_{\phi = \langle \phi^0 \rangle + \langle \phi^1 \rangle}$$

Expanding around $\langle \phi \rangle = \langle \phi^0 \rangle$ and using that $M^2_{\alpha\beta} = 0$ at leading order, as well as:

$$i \sum_b F_{ab} Z_{bn} = - \sum_m T^a_{nm} \langle \phi^0_m \rangle$$

$$(F \cdot Z = v)$$

$$\rightsquigarrow Z_{an} = i \sum_\beta F^{-1}_{a\beta} \, \delta^\beta \langle \phi^0_n \rangle$$

with $\delta^\beta \langle \phi_n \rangle = \sum_m T^\alpha_{nm} \langle \phi^0_m \rangle$, we obtain

$$M^2_{cd} = -\sum_{ab} F^{-1}_{ca} F^{-1}_{db} \Big\{ \delta^{\alpha}\langle\phi^0_n\rangle \, \delta^{b}\langle\phi^0_m\rangle \frac{\partial^2 V_I(\varphi)}{\partial\langle\phi^0_m\rangle\,\partial\langle\phi^0_n\rangle}$$

$$+ \; T^{\alpha} \cdot \delta^{\alpha}\langle\varphi^0_n\rangle \frac{\partial V_I(\varphi)}{\partial\langle\phi_n\rangle} \Big\}$$

We can prove that this is positive definite.

In the following we will ~~demo~~ tie the loose ends of the discussion of Spontaneous Breaking of approximate symmetries:

1. We can find a solution of the vacuum alignment condition
2. The Goldstone - boson mass matrix is positive definite.

Proof of 1

The vacuum alignment condition reads

$$\sum_n \left(\sum_m T^\alpha_{nm} \langle \phi^0_m \rangle \right) \frac{\partial V_I}{\partial \langle \phi^0_n \rangle} = 0 .$$

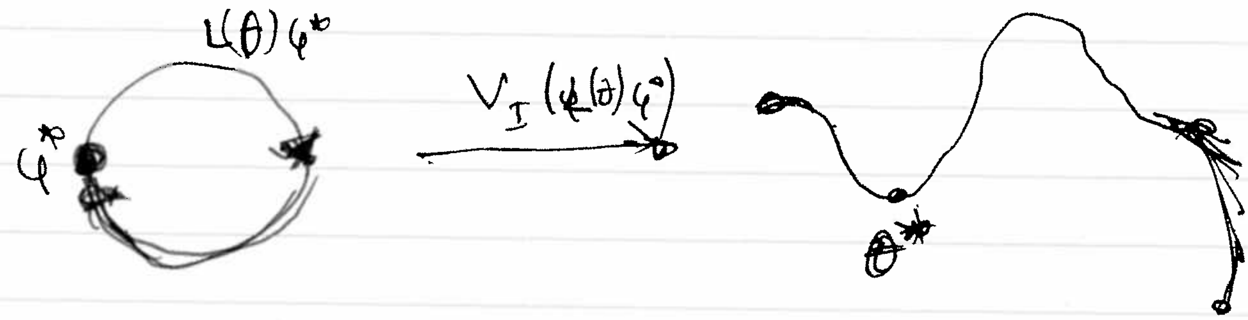
$V_0(\varphi)$ is ~~symme~~ invariant under group transformation

$$\varphi \rightarrow L(\theta) \cdot \varphi .$$

Assume φ_* : $V_0(\varphi_*)$ is a minimum.

Then, also $V_0(L(\theta)\varphi_*)$ corresponds to a minimum.

Consider the perturbation $V_I(L(\theta)\varphi_*)$ as a function of the group parameters θ^α. For compact groups $L(\theta)\varphi_*$ spans a compact manifold, and if V_I is a continuous function then ~~it must~~ $V_I(L(\theta)\varphi^*)$ must have a minimum.



At the minimum,

$$0 = \frac{\partial V_I(L(\theta)\varphi^*)}{\partial \theta^\alpha} = \sum_\eta \frac{\partial V_I(\varphi)}{\partial \varphi_\eta}\Bigg|_{\varphi = L(\theta)\varphi^*} \cdot$$

$$\cdot \frac{\partial (L(\theta)\varphi^*)_\eta}{\partial \theta^\alpha}$$

Recall that

$$L(\theta) = e^{iT^\alpha \theta^\alpha}$$

$$\rightsquigarrow \frac{\partial L(\theta)}{\partial \theta^\beta} = (iT^\beta)\, L(\theta).$$

and thus:

$$0 = \sum_\eta \frac{\partial V_I}{\partial \langle L(\theta^*)\varphi^*\rangle} T^\alpha_{\eta m} \left(L(\theta^*)\varphi^*\right)_m$$

This is the vacuum alignment condition

$\langle \phi \rangle = L(\theta^*)\varphi^*$ satisfies it.

To recap, in order to find which of the vacuum states is chosen for an approximate symmetry which is spontaneously broken we need ~~to~~:

α) Find all vev's which minimize the "exact-symmetry" potential $V_0(\varphi)$. These are degenerate

$$L(\theta) \langle \phi^* \rangle \quad : \quad 0 = \frac{\partial V_0}{\partial \varphi}\Big|_{\varphi = L(\theta)\langle\varphi^*\rangle}$$

β) Find which value of the symmetry group parameters minimize also the symmetry breaking perturbation

$$\theta^* : \quad V_I \left(L(\theta^*) \langle \varphi^* \rangle \right) \text{ is minimum.}$$

The true vacuum state is the one with a field vev

$$\langle \phi \rangle \equiv L(\theta^*) \langle \phi^* \rangle .$$

The recipe is very intuitive. Among the degenerate vacuum states of the symmetry-exact theory, we select the one which minimizes the perturbation of the potential.

Proof of 2

The mass matrix is

$$M^2_{cd} = -\sum_{ab} F^{-1}_{c\alpha} F^{-1}_{db} \cdot \left\{ \delta^a \langle \phi^0_n \rangle \delta^b \langle \phi^0_m \rangle \frac{\partial^2 V_I(\varphi)}{\partial \langle \phi^0_n \rangle \partial \langle \phi^0_m \rangle} + T^\alpha \delta^a \langle \phi^0_n \rangle \cdot \frac{\partial V_I}{\partial \langle \phi^0_n \rangle} \right\}$$

Using that $\langle \phi^0 \rangle = L(\theta^*) \langle \varphi^* \rangle$

we obtain:

$$M^2_{\alpha b} = \sum_{cd} F^{-1}_{ac} F^{-1}_{bd} \frac{\partial^2 V_I(L(\theta)\varphi_*)}{\partial \theta_c \partial \theta_d}$$

Given that $V_I(L(\theta)\varphi_*)$ is a minimum, of the perturbation, $M^2_{\alpha b}$ is positive definite. We shall see examples of it.

The Goldstone-boson particles of a spontaneously broken approximate symmetry are massive. They are termed "pseudo-Goldstone bosons". The phenomenon occurs in QCD. It is also used to solve the hierarchy problem in Electroweak theory.