Sheet 1

Deadline: 10 October 2011

Exercise 1 [*Lagrange density of electrodynamics*]:
(i) Consider the Lagrange density

$$\mathcal{L}_0[A_\mu] = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu \ , \qquad \text{where} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ ,$$

and j^μ is some external source field. Show that the Euler-Lagrange equations are the inhomogenous Maxwell equations. (The homogeneous Maxwell equations $\partial_\lambda F_{\mu\nu}$+cykl.=0 are automatically satisfied provided that the fields are expressed in terms of the potentials A_μ. The usual electromagnetic fields are defined by $E^i = -F^{0i}$ and $\epsilon^{ijk}B^k = -F^{ij}$.)

(ii) Construct the energy-momentum tensor for this theory using the general formula

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\partial^\nu \phi_i - g^{\mu\nu}\mathcal{L} \ , \tag{1}$$

where ϕ_i labels the different fields, and the sum over i is implicit. Assuming that there are no external charges, $j^\mu = 0$, show that the resulting tensor is indeed conserved but not symmetric. In order to make it symmetric consider

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} \ , \tag{2}$$

where $K^{\lambda\mu\nu}$ is anti-symmetric in the first two indices. Show that $\hat{T}^{\mu\nu}$ is conserved provided that $T^{\mu\nu}$ is conserved. By taking

$$K^{\lambda\mu\nu} = F^{\mu\lambda}A^\nu \tag{3}$$

show that the modified stress energy tensor $\hat{T}^{\mu\nu}$ is symmetric, and that it leads to the standard formulae for the electromagnetic energy and momentum densities

$$\mathcal{E} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) \ , \qquad \mathbf{S} = \mathbf{E} \wedge \mathbf{B} \ . \tag{4}$$

Exercise 2 [*Propagators*]:
(i) The dynamical equations of motion of many field theories are of the form

$$\left(\Box + m^2\right) \phi(x) = j(x) \ , \tag{5}$$

where j may be a function of the field ϕ. In many situations it is useful to find standard solutions that satisfy (5) with j replaced by a delta function,

$$\left(\Box_x + m^2\right) G(x,y) = \delta^{(4)}(x-y) \ , \tag{6}$$

and that obey suitable boundary conditions; the corresponding solutions are usually referred to as *Green's functions*. Assuming that the boundary conditions are translation invariant, $G(x,y) \equiv G(x-y)$, show that the solution of (6) is given by

$$G(x-y) = \frac{1}{(2\pi)^4}\int d^4p \, e^{-ip\cdot(x-y)} \, \tilde{G}(p) \ , \tag{7}$$

provided that $\tilde{G}(p)$ satisfies

$$(-p^2 + m^2)\,\tilde{G}(p) = 1\,. \tag{8}$$

(ii) In order to actually solve for the Green's function $G(x, y)$ we have to explain how the singularity on the mass shell $p^2 = m^2$ is treated in the integral. The standard way to proceed is to deform the integration contour for p^0 away from the real axis. Two important examples are

$$\tilde{G}_{\rm ret}(p) = \frac{-1}{(p_0 + i\epsilon)^2 - \mathbf{p}^2 - m^2}\,, \qquad \tilde{G}_{\rm adv}(p) = \frac{-1}{(p_0 - i\epsilon)^2 - \mathbf{p}^2 - m^2}\,, \tag{9}$$

where $\epsilon > 0$ is taken to zero once the integral in (7) has been performed. Denoting the corresponding Green's functions by $G_{\rm ret}(x)$ and $G_{\rm adv}(x)$, respectively, show that we have $G_{\rm ret}(x) = 0$ for $x^0 < 0$, and $G_{\rm adv}(x) = 0$ for $x^0 > 0$.

(iii) Show that (up to terms of order ϵ^2), $G_{\rm ret}(x)$ and $G_{\rm adv}(x)$ are Lorentz invariant, and hence deduce that $G_{\rm ret}(x) = 0$ for all x outside the forward light-cone, and $G_{\rm adv}(x) = 0$ for all x outside the backward light-cone.

(iv) Evaluate $G_{\rm ret}$ explicitly, and show that it has the form

$$G_{\rm ret}(x) = i\frac{\theta(x^0)}{(2\pi)^3}\int \frac{d^3\mathbf{p}}{2\omega_p}\Big\{e^{-i\omega_p x^0 + i\mathbf{p}\cdot\mathbf{x}} - e^{i\omega_p x^0 + i\mathbf{p}\cdot\mathbf{x}}\Big\}\,, \tag{10}$$

where $\omega_p = \sqrt{\mathbf{p}^2 + m^2}$. This answer has a direct physical interpretation: it is the difference of two plane wave solutions of the Klein-Gordon equation; the first has positive frequency and corresponds to the propagation of particles, while the second has negative frequency and corresponds to the propagation of anti-particles.

(v) In quantum field theory another propagator plays an important role, the so-called Feynman propagator

$$G_F(x) = -\frac{1}{(2\pi)^4}\int d^4p\, e^{-ip\cdot x}\,\frac{1}{p^2 - m^2 + i\epsilon}\,. \tag{11}$$

Show that G_F is symmetric, $G_F(-x) = G_F(x)$ for $\epsilon \to 0$, and do the p^0 integration to deduce that

$$G_F(x) = \frac{i}{(2\pi)^3}\int \frac{d^3\mathbf{p}}{2\omega_p}\Big\{\theta(x^0)e^{-i\omega_p x^0 + i\mathbf{p}\cdot\mathbf{x}} + \theta(-x^0)e^{i\omega_p x^0 + i\mathbf{p}\cdot\mathbf{x}}\Big\}\,. \tag{12}$$

Does this propagator vanish outside the forward or backward lightcone?