Sheet 8

Deadline: 28 November 2011

Exercise 1 [*Discrete Symmetries*]:
In this question we shall construct the discrete symmetry transformations acting on the Dirac field, whose mode expansion is of the form

$$\psi(x) = \sum_{\alpha=1,2} \int d\tilde{k} \left(b_\alpha(k) u^{(\alpha)}(k) e^{-ik\cdot x} + d_\alpha^\dagger(k) v^{(\alpha)}(k) e^{ik\cdot x} \right) . \tag{1}$$

We shall work in the chiral basis for the γ matrices, for which the normalised spinors can be written as

$$u^\alpha(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^\alpha \\ \sqrt{p\cdot\bar\sigma}\,\xi^\alpha \end{pmatrix} \qquad \text{and} \qquad v^\alpha(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^\alpha \\ -\sqrt{p\cdot\bar\sigma}\,\xi^\alpha \end{pmatrix} . \tag{2}$$

Here ξ is a two-component spinor normalised to unity (compare Sheet 7, Ex 3).

(i) The *parity transformation* P acts on the position variables x as $x = (x^0, \mathbf{x}) \mapsto \bar{x} \equiv (x^0, -\mathbf{x})$. We want to define an action of P on the Fock space so that

$$P\,\psi(t,\mathbf{x})\,P^{-1} = U(P)\,\psi(t,-\mathbf{x}) . \tag{3}$$

Show that this identity is satisfied with $U(P) = \gamma^0$ if we define the action of P on the modes as

$$P\,b_\alpha(k)\,P^{-1} = b_\alpha(\bar{k}) , \qquad P\,d_\alpha^\dagger(k)\,P^{-1} = -d_\alpha^\dagger(\bar{k}) , \tag{4}$$

where $\bar{k} = (k^0, -\mathbf{k})$.

Hint: Note that we have $p\cdot x = \bar{x}\cdot\bar{p}$ and show that

$$u^\alpha(p) = \gamma^0\,u(\bar{p}) , \qquad v^\alpha(p) = -\gamma^0\,v(\bar{p}) . \tag{5}$$

(ii) Similarly, check that the *time reversal transformation* T, mapping $x = (x^0, \mathbf{x}) \mapsto \tilde{x} \equiv (-x^0, \mathbf{x})$, acts as

$$T\,\psi(t,\mathbf{x})\,T^{-1} = \gamma^1\gamma^3\,\psi(-t,\mathbf{x}) , \tag{6}$$

provided that it acts on the modes as

$$T\,b_\alpha(k)\,T^{-1} = \epsilon_{\alpha\beta}\,b_\beta(\bar{k}) , \qquad T\,d_\alpha^\dagger(k)\,T^{-1} = \epsilon_{\alpha\beta}\,d_\beta^\dagger(\bar{k}) , \tag{7}$$

where $\epsilon_{12} = -\epsilon_{21} = 1$ and $\epsilon_{11} = \epsilon_{22} = 0$.

Hint: Note that T is *anti-unitary* and reverses the spins, $\xi_\alpha \to \xi'_\alpha \equiv \epsilon_{\alpha\beta}\xi_\beta$. Using the identities $\boldsymbol{\sigma}^*\sigma^2 = -\sigma^2\boldsymbol{\sigma}$ and $\xi'_\alpha = -i\sigma^2\xi^*_\alpha$, prove that

$$\epsilon_{\alpha\beta}(u^\beta(\bar{p}))^* = -\gamma^1\gamma^3 u^\alpha(p) , \qquad \epsilon_{\alpha\beta}(v^\beta(\bar{p}))^* = -\gamma^1\gamma^3 v^\alpha(p) . \tag{8}$$

Exercise 2 [*LSZ reduction formula for fermions*]:
Prove the LSZ reduction formula for fermions

$$
\begin{aligned}
&{}_{\text{out}}\langle \cdots d(q_i') \cdots b(q_i) \cdots | \cdots b^\dagger(k_i) \cdots d^\dagger(k_i') \cdots \rangle_{\text{in}} \\
&\quad = \text{disconnected terms} \\
&\qquad + (-iZ_2^{-1/2})^{n'}\,(iZ_2^{-1/2})^{n} \int d^4x_i \cdots d^4y_i' \cdots \\
&\qquad \times \exp\left(-i\sum_i k\cdot x + k'\cdot x' - q\cdot y - q'\cdot y'\right)\cdots \\
&\qquad \times \bar{u}(q_i)\,(i\,\overrightarrow{\not\partial}_{y_i} - m)\cdots \bar{v}(k_i')\,(i\,\overrightarrow{\not\partial}_{x_i'} - m) \\
&\qquad \times \langle 0|\mathcal{T}\left[\cdots \bar{\psi}(y_i')\cdots \psi(y_i)\,\bar{\psi}(x_i)\cdots \psi(x_i')\cdots\right]|0\rangle \\
&\qquad \times (-i\,\overleftarrow{\not\partial}_{x_i} - m)\,u(k_i)\cdots(-i\,\overleftarrow{\not\partial}_{y_i'} - m)\,v(q_i')\cdots\ ,
\end{aligned}
\tag{9}
$$

where the in-coming particles and anti-particles have momenta labelled by $(k_1, \ldots)$ and $(k_1', \ldots)$, respectively, while $(q_1, \ldots)$ and $(q_1', \ldots)$ are the momenta of the out-going particles and anti-particles, respectively. The corresponding space-time variables are denoted by x, x', y und y', respectively, while n and n' denote the total number of particles and anti-particles, respectively.

Hint: Consider, as in the lecture, the recursion step in which $b_{\text{in}}^\dagger(k_1)$ and $b_{\text{out}}(q_1)$ (or $d_{\text{in}}^\dagger(k_1')$ and $d_{\text{out}}(q_1')$) are removed. The factor Z_2 is defined via

$$
\psi \overset{t\to\infty}{\longrightarrow} Z_2^{1/2}\psi_{\text{out}}\ , \qquad \psi \overset{t\to-\infty}{\longrightarrow} Z_2^{1/2}\psi_{\text{in}}\ ,
$$

and we use the mode expansions

$$
\psi_{\text{in,out}}(x) = \sum_{\alpha=1,2}\int \frac{d^3\mathbf{p}}{(2\pi)^3\,2p^0}\left[b_{\text{in,out}\,\alpha}(p)\;u^{(\alpha)}(p)\;e^{-ip\cdot x} \;+\; d_{\text{in,out}\,\alpha}^\dagger(p)\;v^{(\alpha)}(p)\;e^{ip\cdot x}\right]
$$

$$
\begin{aligned}
b_{\text{out}\,\alpha}(k) &= \int_t d^3\mathbf{x}\;\bar{u}^{(\alpha)}(k)\,\gamma^0\,e^{ik\cdot x}\;\psi_{\text{out}}(x) \\
d_{\text{in}\,\alpha}^\dagger(k) &= \int_t d^3\mathbf{x}\;\bar{v}^{(\alpha)}(k)\,\gamma^0\,e^{-ik\cdot x}\;\psi_{\text{in}}(x) \\
b_{\text{in}\,\alpha}^\dagger(k) &= \int_t d^3\mathbf{x}\;\bar{\psi}_{\text{in}}(x)\,\gamma^0\,u^{(\alpha)}(k)\,e^{-ik\cdot x} \\
d_{\text{out}\,\alpha}(k) &= \int_t d^3\mathbf{x}\;\bar{\psi}_{\text{out}}(x)\,\gamma^0\,v^{(\alpha)}(k)\,e^{ik\cdot x}\ ,
\end{aligned}
$$

as well as analogous identities for which the roles of *in* and *out* are interchanged. Note that these identities hold for arbitrary times t, and use

$$
\left(\lim_{t\to\infty} - \lim_{t\to-\infty}\right)\int d^3\mathbf{x}\,F(\mathbf{x},t) = \lim_{\substack{t_f\,\to\,\infty \\ t_i\,\to\,-\infty}} \int_{t_i}^{t_f} dt\,\frac{\partial}{\partial t}\int d^3\mathbf{x}\,F(\mathbf{x},t)
$$

to express the space integrals in term of space-time integrals.