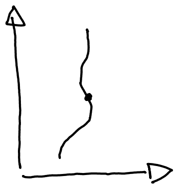
2 Relativistic Point Particle

Let us start slowly with something else: a relativistic particle. Here we will encounter several issues of string theory, but in a more familiar setting. There are many formulations, we will discuss several.

2.1 Non-Relativistic Actions

First: a free non-relativistic point particle $\vec{x}(t)$. Action with resulting equations of motion (e.o.m.):

$$S[\vec{x}] = \int dt\, \tfrac{1}{2} m \dot{\vec{x}}(t)^2, \qquad \ddot{\vec{x}}(t) = 0.$$



Momentum and energy from Hamiltonian formulation:

$$\vec{p}(t) = \frac{\partial S[\vec{x}]}{\partial \dot{\vec{x}}(t)} = m\dot{\vec{x}}(t), \qquad E(t) = H(t) = \frac{\vec{p}(t)^2}{2m}\,.$$

Note: functional variation removes the integral.

Promote the above to a relativistic particle:

$$S = -\int dt\, mc\sqrt{c^2 - \dot{\vec{x}}^2} \approx \int dt\, \left(-mc^2 + \tfrac{1}{2} m\dot{\vec{x}}^2 + \tfrac{1}{8} mc^{-2}\dot{\vec{x}}^4\right).$$

Derive e.o.m.:

$$(c^2 - \dot{\vec{x}}^2)\ddot{\vec{x}} + (\dot{\vec{x}} \cdot \ddot{\vec{x}})\dot{\vec{x}} = 0.$$

They imply collinearity $\ddot{\vec{x}} = \alpha\dot{\vec{x}}$. Substitute to get $\alpha c^2 \dot{\vec{x}} = 0$ hence $\ddot{\vec{x}} = 0$. Momentum and energy read

$$\vec{p} = \frac{mc\dot{\vec{x}}}{\sqrt{c^2 - \dot{\vec{x}}^2}}\,, \qquad E = c\sqrt{m^2c^2 + p^2}\,.$$

Fine, but is not manifestly relativistic: Non-relativistic formulation of a relativistic particle. Want a manifestly relativistic formulation using 4-vectors $X^\mu = (ct, \vec{x})$ and $P_\mu = (E/c, \vec{p})$. Let us set $c = 1$ for convenience from now on.

- Momentum P_μ is already a good 4-vector:

  $$P^2 = -E^2 + \vec{p}^2 = -m^2.$$

  Mass shell condition $P^2 = -m^2$ manifestly relativistic. But origin/role of $\vec{p}$ and E is quite distinct.

- Position $X^m(t) = (t, \vec{x}(t))$ and action make explicit reference to time t (in a particular Lorentz frame)

$$S = -\int dt\, m\sqrt{-\left(\frac{dX(t)}{dt}\right)^2}.$$

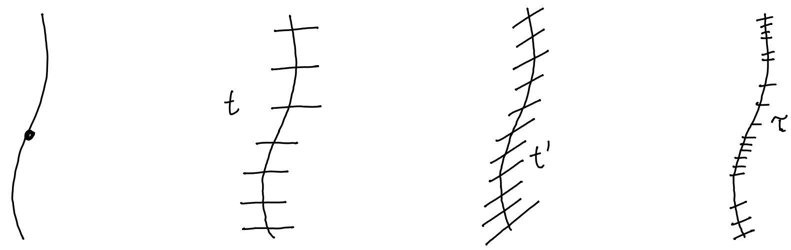
- Note that Hamiltonian framework distinguishes between space and time: Explicit reference to time derivatives.

2.2 Worldline Action

Notice: above action measures Lorentz-invariant proper time s of the particle's path $X^\mu(t)$ in spacetime (worldline)

$$S = -m \int ds, \quad \text{where } ds^2 = -dX^2.$$

Proper time depends only on the location of the worldline, but not on a particular Lorentz frame (definition of t) or parametrisation of the worldline (through t).



Let us assume an arbitrary parametrisation $X^\mu(\tau)$ of the worldline through some curve parameter τ. The proper time action reads (now dot denotes $d/d\tau$)

$$S = -\int d\tau\, m\sqrt{-\left(\frac{dX(\tau)}{d\tau}\right)^2} = -\int d\tau\, m\sqrt{-\dot{X}^2}\,.$$

Nice manifestly relativistic formulation. Notice: 4 undetermined functions $X^\mu(\tau)$ instead of 3 undetermined functions $\vec{x}(t)$; new function $t(\tau)$.

Use this as starting point, derive equations of motion

$$\dot{X}^2\ddot{X}^\mu = (\dot{X}\cdot\ddot{X})\dot{X}^\mu.$$

Implies collinearity $\ddot{X}^\mu = c\dot{X}^\mu$ for all τ with variable $c(\tau)$. Meaning: worldline straight. ✓

Next derive momenta as derivatives of S w.r.t. $\dot{X}^\mu$

$$P_\mu = \frac{m\dot{X}_\mu}{\sqrt{-\dot{X}^2}}\,.$$

Obey mass shell condition $P^2 = -m^2$! ✓

Only three independent P_μ but four independent X^μ!
Moreover naive Hamiltonian is strictly zero $H = 0$!
Signals presence of constraint and gauge invariance:

- Reparametrizing $\tau' = f(\tau)$ has no effect on physics.
- Redundancy of description: worldline coordinate τ.
- One linear dependency among the e.o.m. for X^μ.
- Gauge invariance effectively removes one X^μ, e.g. time $t(\tau)$.
- Situation inconvenient for Hamiltonian framework/QM.
- Better to fix a gauge, many choices, pick a convenient one.

Obtained a fully relativistic formulation, but packaged with complication of gauge invariance. In fact, gauge invariance often considered a virtue: Symmetry!

Above worldline action has two further drawbacks:

- Is non-polynomial; inconvenient for quantisation.
- Does not work for massless particles $m = 0$.

2.3 Polynomial Action

There is an equivalent action with an auxiliary variable $e(\tau)$

$$S = \int d\tau \left(\tfrac{1}{2} e^{-1} \dot{X}^2 - \tfrac{1}{2} e m^2 \right).$$

The resulting e.o.m. read

$$m^2 e^2 + \dot{X}^2 = 0, \qquad e\ddot{X}^\mu - \dot{e}\dot{X}^\mu = 0.$$

In combination they yield the same old equation for X^μ. The momentum conjugate to X^μ reads $P_\mu = e^{-1}\dot{X}_\mu$, hence the equation of motion for e reduces to $P^2 = -m^2$. Momentum conjugate to e vanishes signalling a constraint.

Massless case $m = 0$ works at every step of above derivation, yields constant $P^\mu = e^{-1}\dot{X}^\mu$ as well as $P^2 = 0$. Notice: e not fixed by e.o.m.; commonly gauge freedom remains for massless case.

Field e has a nice geometrical interpretation: Einbein specifying a metric $g_{\tau\tau} = -e^2$ on the worldline. All terms in the action are in the right combination; remain invariant under changing worldline coordinates (e transforms according to $e' = e\, d\tau'/d\tau$).

Here einbein e is a dynamical variable. Curiously, e.o.m. picks out metric induced by ambient space. When substituting solution for e recover action of Sec. 2.2.

2.4 Various Gauges

We have freedom to fix one of the coordinates $X^\mu(\tau)$ at will. Some more or less useful choices:

- **Temporal Gauge.** $t(\tau) = \tau$ or $t(\tau) = \alpha\tau$.
  Reduces to non-relativistic treatment of Sec. 2.1.
- **Spatial Gauge.** $z(\tau) = \alpha\tau$.
  Works locally except at turning points of $z(\tau)$.
- **Lightcone Gauge.** $x^+(\tau) := t(\tau) + z(\tau) = \alpha\tau$.
  Useful in some cases; prominent in string theory.
- **Proper Time Gauge.** $ds = d\tau$.
  Fixes $t(\tau)$ through integral
  $$t(\tau) = \int^\tau d\tau' \sqrt{1 + \dot{\vec{x}}(\tau')^2}\,.$$
  Action becomes trivial $S = -\int d\tau$; deal with constraint.
- **Constant Einbein.** $\dot{e} = 0$.
  In polynomial formulation, gauge fixing may involve e. Customary gauge choice is constant e. E.o.m. reduces to
  $$\ddot{X} = 0.$$
  We replace dynamical variable e by a constant; we must remember its equation of motion
  $$\dot{X} + m^2 e^2 = 0.$$
  In gauge fixed formulation it becomes constraint.

2.5 Quantisation

Quantisation can be done in several different ways. Depends on the choice of classical formulation. Let us pick polynomial action discussed in Sec. 2.3. For the Hamiltonian formulation it is best to also fix a gauge; we will choose the einbein e to be constant. Momenta P associated to X and resulting Hamiltonian read:

$$P = e^{-1}\dot{X}, \qquad H = \tfrac{1}{2}e\big(P^2 + m^2\big).$$

Conventionally, a state $|\Psi\rangle$ is given by a wave function of position variables and time

$$|\Psi\rangle = \int d^4X\, \Psi(X,\tau)\, |X\rangle.$$

Slightly more convenient to immediately Fourier transform to momentum space $|X\rangle \simeq \int d^4P\, e^{iP\cdot X}\, |P\rangle$

$$|\Psi\rangle = \int d^4P\, \Psi(P,\tau)\, |P\rangle, \qquad \Psi(P,\tau) = \int d^4X\, e^{iP\cdot X}\Psi(X,\tau).$$

Schrödinger equation reads

$$i\dot{\Psi} = H\Psi = \tfrac{i}{2}e\big(P^2 + m^2\big)\Psi.$$

Obviously, solved by

$$\Psi(P,\tau) = \exp\big(-\tfrac{i}{2}e(P^2 + m^2)\tau\big)\Phi(P).$$

Need to remember that system is constrained; wave function must vanish whenever constraint not satisfied:

$$(P^2 + m^2)\Psi(P, \tau) = 0.$$

In effect, physical states $\Psi(P, \tau) = \Phi(P)$ are independent of τ. Makes perfect sense: worldline coordinate τ unphysical. Schrödinger equation governing τ-evolution replaced by constraint $P^2 + m^2 = 0$ (governing t-evolution).

Fourier transform wave function back to position space $\Phi(X)$; constraint becomes the Klein–Gordon equation for spin-0 field

$$(-\partial^2 + m^2)\Phi(X) = 0.$$

2.6 Interactions

Obviously, free particle is easy; eventually would like to include interactions. Let us sketch how to add interactions with external potentials and with other particles:

Electrical and Gravitational Fields. Coupling to electrical and gravitational fields takes a very geometric form

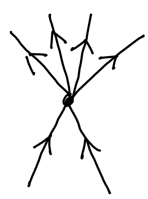
$$S = \int d\tau \left(\tfrac{1}{2}e^{-1}g_{\mu\nu}(X)\dot{X}^\mu\dot{X}^\nu - \tfrac{1}{2}em^2 + A_\mu(X)\dot{X}^\mu\right).$$

A_μ potential for the electromagnetic field $F_{\mu\nu} = \partial_\mu A_\nu + \partial_\nu A_\mu$. Likewise $g_{\mu\nu}$ is the gravitational potential; takes the form of the metric of a curved spacetime.

These are fixed external fields: Unaffected by presence of particle, but influence its motion. Note: Fields are evaluated at dynamical position $X^\mu(\tau)$.

In quantum mechanics, one usually assumes weak interactions. Formally allows to work with free quantum fields; interactions are introduced in perturbative fashion. When free particle enters potential field, it scatters off of it. Dominant contribution from single scattering; multiple interactions suppressed. Only in rare instances, potentials can be handled exactly.

Interactions Among Particles. Local interactions: Several particles meet at some spacetime point and split up, potentially into a different number of particles. In worldline formulation achieved by introducing vertices where several particle worldlines meet:



This is not standard treatment of interaction. Typically interaction of n fields: term Φ^n in QFT action. Our method is not very convenient, but it works as well.

Nevertheless instructive, mimics Feynman rules; It is the standard procedure for string theory.

2.7 Conclusions

- Seen many equivalent formulations of same physical system.
- Had to deal with gauge invariance and constraints.
- Different number of degrees of freedom (d.o.f.), but number of solutions (modulo gauge) always the same.
- Quantised the free relativistic particle.
- Discussed interactions.
- Not always most convenient path chosen; but treatment of string will be analogous.