# 9 String Backgrounds

Have seen that string spectrum contains graviton. Graviton interacts according to laws of General Relativity. General Relativity is a theory of spacetime geometry. Strings can move in curved backgrounds.

How are strings and gravity related?

- Should we quantise the string background?
- Is the string graviton the same as the Einstein graviton?
- Is there a backreaction between strings and gravity?

## 9.1 Graviton Vertex Operator

Compare graviton as string excitation and background. Assume momentum $q$ and polarisation $\epsilon_{\mu\nu}$.

**Vertex Operator Construction.** Graviton represented by closed string state

$$|\epsilon; q\rangle = \epsilon_{\mu\nu}\big(\alpha^{\mathrm{L},\mu}_{-1}\alpha^{\mathrm{R},\nu}_{-1} + \alpha^{\mathrm{L},\nu}_{-1}\alpha^{\mathrm{R},\mu}_{-1}\big)|0; q\rangle.$$

Corresponding vertex operator reads

$$\begin{aligned}\mathcal{O}^{\mu\nu} &= {:}(\partial X^\mu \bar\partial X^\nu + \partial X^\nu \bar\partial X^\mu)e^{iq\cdot X}{:}\\ &\sim {:}\sqrt{\det -g}\, g^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta X^\nu\,e^{iq\cdot X}{:}.\end{aligned}$$

Insertion into string worldsheet

$$V = \int d^2\xi\,\tfrac{1}{2}\epsilon_{\mu\nu}\mathcal{O}^{\mu\nu}.$$

**Background Metric Construction.** Flat background with plane wave perturbation

$$G_{\mu\nu}(x) = \eta_{\mu\nu} + \epsilon_{\mu\nu}e^{iq\cdot x} + \ldots.$$

Strings couple to background by replacement $\eta_{\mu\nu} \to G_{\mu\nu}$

$$S = -\frac{1}{2\pi\kappa^2}\int d^2\xi\sqrt{-\det g}\,g^{\alpha\beta}\,\tfrac{1}{2}G_{\mu\nu}(X)\,\partial_\alpha X^\mu\,\partial_\beta X^\nu.$$

Same replacement $\eta_{\mu\nu} \to G_{\mu\nu}$ in Nambu–Goto action.

Perturbation of metric same as vertex operator

$$S = S_0 - \frac{1}{2\pi\kappa^2}V + \ldots$$

**Conclusion.** Graviton mode of string is the same as wave on background.

Quantum string on flat space contains gravitons. Gravitons introduce curvature and deform flat background. String theory contains quantum gravity. Large deformations away from flat background represented by coherent states of gravitons.

String theory can be formulated on any background. String quantisation probes nearby backgrounds. Low-energy physics depends on classical background. Full quantum string theory is background independent, contains all backgrounds as different states (same as QG).

## 9.2 Curved Backgrounds

Consider strings on a curved background $G_{\mu\nu}(x)$, curious insight awaits. Action in conformal gauge

$$S = -\frac{1}{2\pi\kappa^2}\int d^2\xi\,\tfrac{1}{2}G_{\mu\nu}(X)\,\eta^{\alpha\beta}\partial_\alpha X^\mu\,\partial_\beta X^\nu.$$

For generic metric $G$, e.o.m. for $X$ are non-linear.

Type of model called non-linear sigma model. String background called target space. Metric field $G_{\mu\nu}(x)$ is sigma model coupling. Infinitely many couplings (Taylor expansion of $G$).

In most QFT's couplings are renormalised. Problem here:

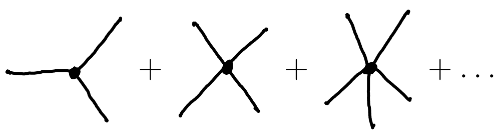
- Classical action has conformal symmetry.
- Conformal symmetry indispensable to remove one d.o.f..
- Renormalised coupling $G(x,\mu)$ depends on scale $\mu$.
- New scale breaks quantum conformal invariance. Anomaly!

**Renormalisation.** Compute the conformal anomaly. Background field quantisation:

- Pick (simple) classical solution $X_0$ of string e.o.m..
- add perturbations $X = X_0 + \kappa Y$. Quantum field $Y$.

Expansion of action $S[X] = S[X_0] + Y^2 + \kappa Y^3 + \ldots$ in orders of $Y$

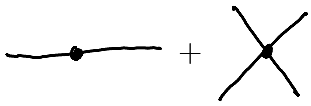
- Value of classical action $S[X_0]$ at $Y^0$ irrelevant.
- No linear term in $Y$ due to e.o.m. for $X_0$.
- Order $Y^2$ is kinetic term for quantum field $Y$. 
- Order $Y^3, Y^4, \ldots$ are cubic, quartic, … interactions



Use target space diffeomorphisms s.t. locally

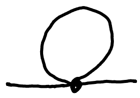
$$S = -\int \frac{d^2\xi}{2\pi}\,\eta^{\alpha\beta}\left(\eta_{\mu\nu}\partial_\alpha Y^\mu\partial_\beta Y^\nu + \tfrac{1}{3}\kappa^2 R_{\mu\rho\nu\sigma}\partial_\alpha Y^\mu\partial_\beta Y^\nu Y^\rho Y^\sigma\right)$$

$R_{\mu\rho\nu\sigma}(x)$ is target space curvature tensor.

Kinetic term and quartic vertex:  $+\ldots$.

At one loop we get tadpole diagram. Insert two-point correlator

$$\kappa^2 R_{\mu\rho\nu\sigma}\partial_\alpha Y^\mu(\xi)\partial_\beta Y^\nu(\xi)\langle Y^\rho(\xi)Y^\sigma(\xi)\rangle$$



but we know for $\xi_1 \to \xi_2$

$$\langle Y^\rho(\xi_1)Y^\sigma(\xi_2)\rangle \simeq -\eta^{\rho\sigma}\log|\xi_1-\xi_2|.$$

Not exact, but UV behaviour fixed by conformal symmetry. Logarithmic singularity responsible for renormalisation. $G_{\mu\nu}$ is running coupling, beta function

$$\frac{\mu\partial G}{\partial\mu} = \beta_{\mu\nu} = \kappa^2 R_{\mu\nu}, \qquad R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}.$$

**Anomaly.** Scale dependence breaks conformal symmetry: Trace of stress energy tensor after renormalisation

$$\eta^{\alpha\beta}T_{\alpha\beta} = -\frac{1}{2\kappa^2}\,\beta_{\mu\nu}\eta^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu.$$

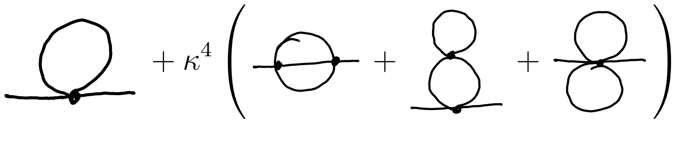
Anomaly of Weyl symmetry! (gauge fixed already)

Conformal/Weyl symmetry is essential for correct d.o.f.. Remove by setting $\beta_{\mu\nu}=0$. Einstein equation!

$$R_{\mu\nu} = 0.$$

Quantum strings can propagate only on Einstein backgrounds. General relativity! Spin-2 particles at level 1 are gravitons.

**Higher Corrections.** There are corrections to the beta function from higher perturbative orders in $\kappa^2$

$\kappa^2$  $+\ldots$

$$\beta_{\mu\nu} = \kappa^2 R_{\mu\nu} + \tfrac{1}{2}\kappa^4 R_{\mu\rho\sigma\kappa}R_\nu{}^{\rho\sigma\kappa} + \ldots\,.$$

Also corrections from the expansion in the string coupling $g_\mathrm{s}$.

Corrections to Einstein equations at Planck scale: $\beta_{\mu\nu}=0$.

## 9.3 Form Field and Dilaton

What about the other (massless) fields? Two-form $B_{\mu\nu}$ and dilaton scalar $\varPhi$?

Two-form couples via antisymmetric combination

$$\frac{1}{2\pi\kappa^2}\int d^2\xi\,\tfrac{1}{2}B_{\mu\nu}(X)\,\varepsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu = \frac{1}{2\pi\kappa^2}\int B.$$

In fact, canonical coupling of two-form to 2D worldsheet. Analogy to charged particle in electromagnetic field. String has two-form charge.

Dilaton couples to worldsheet Riemann scalar

$$\frac{1}{4\pi}\int d^2\xi\,\sqrt{-\det g}\,\varPhi(X)\,R[g].$$

Interesting for several reasons:

- Euler characteristic $\chi$ of the worldsheet appears.
- Not Weyl invariant.
- Scalar can mix with gravity.
- Can get away from 26 dimensions.

**Low-Energy Effective Action.** First discuss the various beta functions (trace of renormalised stress energy tensor $T$)

$$\begin{aligned} g^{\alpha\beta}T_{\alpha\beta} = & -\frac{1}{2\kappa^2}\left(\sqrt{-\det g}\,\beta^G_{\mu\nu}\eta^{\alpha\beta}+\beta^B_{\mu\nu}\varepsilon^{\alpha\beta}\right)\partial_\alpha X^\mu\partial_\beta X^\nu \\ & -\tfrac{1}{2}\beta^\varPhi R[g] \end{aligned}$$

with

$$\begin{aligned} \beta^G_{\mu\nu} &= \kappa^2 R_{\mu\nu} + 2\kappa^2 D_\mu D_\nu\varPhi - \tfrac{1}{4}\kappa^2 H_{\mu\rho\sigma}H_\nu^{\rho\sigma}, \\ \beta^B_{\mu\nu} &= -\tfrac{1}{2}\kappa^2 D^\lambda H_{\mu\nu\lambda} + \kappa^2 D^\lambda\varPhi H_{\mu\nu\lambda}, \\ \beta^\varPhi &= -\tfrac{1}{2}\kappa^2 D^2\varPhi + \kappa^2 D^\mu\varPhi D_\mu\varPhi - \tfrac{1}{24}\kappa^2 H_{\mu\nu\rho}H^{\mu\nu\rho}. \end{aligned}$$

Quantum string consistency requires $\beta^G = \beta^B = \beta^\varPhi = 0$. Standard equations for graviton, two-field and scalar. Follow from an action

$$S \sim \int d^{26}x\,\sqrt{-\det g}\,e^{-2\varPhi}\left(R - \tfrac{1}{2}H_{\mu\nu\rho}H^{\mu\nu\rho} + 4\partial^\mu\varPhi\,\partial_\mu\varPhi\right).$$

String low-energy effective action. Encodes low-energy physics of string theory. Further corrections from curvature and loops.

Trivial solution: $G=\eta$, $B=0$, $\varPhi=\varPhi_0$ (flat background). Can also use torus compactification to reduce dimensions.

**String Coupling.** Suppose $\Phi = \Phi_0$ is constant, then dilaton coupling term is topological

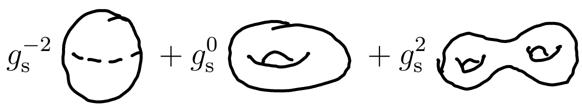
$$\int d^2\xi \sqrt{-\det g}\, R[g] \sim \chi.$$

Measures Euler characteristic $\chi = 2h - 2$ of world sheet.

Set $g_\mathrm{s} = e^{i\Phi_0}$. Then action yields $\chi$ factors of $g_\mathrm{s}$.

$$e^{iS} \simeq e^{i\Phi_0\chi} = g_\mathrm{s}^\chi.$$

Expansion in worldsheet topology.




String coupling $g_\mathrm{s}$ determined through background: Asymptotic value $\Phi_0$ of dilaton field $\Phi$.

**String Frame.** Notice unusual factor of $\exp(-2\Phi)$ in $S$.

Scalar degrees of freedom can mix with metric. Could as well define

$$G'_{\mu\nu} = f(\Phi) G_{\mu\nu}.$$

Remove $\exp(-2\Phi)$ through suitable choice of $f$. Go from “string frame” to “Einstein frame”. Standard kinetic terms for all fields.

**Noncritical Strings.** We have seen earlier that $D \neq 26$ breaks Weyl symmetry. $D$ enters in effective action as worldsheet cosmological constant

$$S = \ldots \bigl(R - \tfrac{1}{2} H_{\mu\nu\rho} H^{\mu\nu\rho} + 4\partial^\mu \Phi\, \partial_\mu \Phi - \tfrac{2}{3}\kappa^{-2}(D - 26)\bigr).$$

Can have $D < 26$, but requires Planck scale curvature.

**Dilaton Scaling.** Dilaton coupling to worldsheet is not Weyl invariant and has unconventional power of $\kappa$.

- Consistent choice.
- Moves classical Weyl breakdown to one loop. Cancel quantum anomalies of other fields.

## 9.4 Open Strings

Open strings lead to additional states, fields and couplings.

- Additional string states; e.g. massless vectors (photon):

$$|\zeta; q\rangle = \zeta_\mu \alpha^\mu_{-1} |0; q\rangle.$$

- Additional vertex operators; e.g. photon

$$V[\zeta, q] \sim \int d\tau\, \zeta_\mu \partial_\tau X^\mu \, \exp(iq \cdot X)$$

- Additional fields to couple to string ends.

Background couplings can be identified as for closed strings. Vertex operator has same effect as background field.

Coupling depends on string boundary conditions: D$p$-brane.

**Neumann Boundaries.** For all coordinates $X^a$, $a = 0, \ldots, p$, with Neumann conditions: couple a one-form gauge field $A$ to end of string

$$\int_{\text{end}} d\tau \dot{X}^a A_a(X) = \int_{\text{end}} A.$$

- natural coupling of a charged point-particle to gauge field.
- string end is a charged point-like object.

Gauge field $A_a$ exists only on D$p$-brane. Okay since string ends constrained to D$p$ brane.

Classical coupling of $A$ respects Weyl symmetry. Quantum anomaly described by beta function

$$\beta^A_a \sim \kappa^4 \partial^b F_{ab}$$

Absence of conformal anomaly requires Maxwell $\partial^b F_{ab} = 0$. Associated low-energy effective action

$$S \sim -\kappa^4 \int d^{p+1}x \, \tfrac{1}{4} F_{ab} F^{ab}.$$

For planar D$p$-brane can also include higher corrections in $\kappa$. Born–Infeld action:

$$S \sim \int d^{p+1}x \sqrt{-\det(\eta_{ab} + 2\pi\kappa^2 F_{ab})}.$$

Leading order is Maxwell kinetic term. Corrections at higher orders in $\kappa$.

**Dirichlet Boundaries.** Coupling of Dirichlet directions $X^m$, $m = p+1, \ldots, D-1$, different.

- $X^m$ fixed, but $X'^m$ can be used.
- Dual field $Y_m$ describes transverse D$p$-brane displacement.
- D$p$-branes are dynamical objects!

Beta function at leading order: massless scalar

$$\beta_a \sim \partial^m \partial_m Y_a$$

Effective action for higher orders: Dirac–Born–Infeld action

$$S \sim \int d^{p+1}x \sqrt{-\det(g_{ab} + 2\pi\kappa^2 F_{ab})}.$$

Induced WS metric $g_{ab} = \partial_a Y^\mu \, \partial_b Y_\mu$. Embedding coordinates $Y$ for D$p$-brane. Combination of

- Dirac action for $p$-branes and
- Born–Infeld action for gauge fields.

**D-Branes in a Curved Background.** Can even add effect of close string fields.

$$S \sim \int d^{p+1}x \, e^{-\Phi} \sqrt{-\det(g_{ab} + 2\pi\kappa^2 F_{ab} + B_{ab})}.$$

- $g_{ab}$ is induced metric from curved background.
- $B_{ab}$ is pull back of 2-form field $B_{\mu\nu}$ to D$p$-brane.
- combination $2\pi\alpha' F_{ab} + B_{ab}$ is gauge invariant.
- dilaton couples as prefactor like for closed string.

**Coincident Branes.** For $N$ coincident branes gauge group enlarges from $U(1)^N$ to $U(N)$.

Gauge field should couple via Wilson line

$$\mathrm{T}\exp \int_{\text{end}} A.$$

Resulting effective action at leading order is

$$S \sim \int d^{p+1}x \, \mathrm{tr}\left(-\tfrac{1}{4}(F_{ab})^2 + \tfrac{1}{2}(D_a Y_m)^2 + \tfrac{1}{4}[Y_m, Y_n]^2\right).$$

Yang–Mills, massless adjoint scalars, quartic interactions.

## 9.5 Two-Form Field of a String

We have seen that strings couple to various fields. A string also generates a field configuration. Analogy: charged point particle generates Coulomb potential.

**Fundamental String.** Consider an infinite straight string along $0, 1$ directions: 1-brane. Generates a two-form potential

$$B = (f^{-1} - 1) dx^0 \wedge dx^1$$

Interactions with metric $G$ and dilaton $\Phi$ require

$$ds^2 = f^{-1} ds_2^2 + ds_{D-2}^2, \qquad e^{2\Phi} = f^{-1}.$$

The function $f$ with $r^2 = x_2^2 + \ldots + x_{D-1}^2$ reads

$$f = 1 + \frac{g_s^2 N \kappa^{D-4}}{r^{D-4}}.$$

This satisfies the low-energy effective string e.o.m. because $f$ is a harmonic function.

Note: Source at the location of the string ($r = 0$).

- E.o.m. follow from combination of spacetime action and worldsheet coupling to two-form

$$\int_D H \wedge *H + \int_2 B$$

Source term $\delta^{D-2}(r)$ absorbed by worldsheet.
- Charge of string measured by Gauss law via $*H$. Put $(D-3)$-dimensional sphere at fixed $r$.

$$Q = \int_{D-3} *H = N.$$

Above string has $N$ units of charge (quantised).

The fundamental string is not a D1-brane: Open strings do not end on it. It is the string itself.

Solutions with more than one centre permissible.

**Magnetic Brane.** Another solution of the string effective e.o.m. describes a $(D-5)$-brane. It uses a dual $(D-4)$-form potential $C$ defined through

$$H = dB, \qquad *H = dC.$$

It carries magnetic charge

$$Q = \int_3 H.$$

The source is located on the $(D-5)$-brane(s). The coupling of $(D-5)$-branes to $C$ compensates source.