Introduction to String Theory — Problem Set 1

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1. On the importance of quantum gravity (easy)

Let us get some intuition on the order of magnitudes:

a) Consider a gravitational atom, an electron bound to a neutron by the gravitational force. Electromagnetic dipole effects can be neglected. Perform a semiclassical calculation to determine the radius of the orbit of the electron (first Bohr radius). Relate this radius to an appropriate distance in physics.

b) In "natural units", where $\hbar$, G and c are set to 1, a stellar black hole radiates like a black body at a temperature given by $kT = 1/8\pi M$. Give the temperature in SI units (reinsert G, $\hbar$ and c) and calculate the temperature of a black hole weighing one solar mass.

2. Relativistic point particle (intermediate)

The action of a relativistic point particle is given by

$$S_{\text{rp}} = -\alpha \int_{\mathcal{P}} ds$$

with the relativistic line element

$$ds^2 = -\eta_{\mu\nu} dX^\mu dX^\nu = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

and α a (yet to be determined) constant. The path $\mathcal{P}$ between two points X_1^μ and X_2^μ can be parametrised by a parameter τ. The integral over the line element ds becomes an integral over the parameter

$$S_{\text{rp}} = -\alpha \int_{\tau_1}^{\tau_2} d\tau \sqrt{-\eta_{\mu\nu} \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \tau}} \,. \qquad (*)$$

a) Parametrise the path by the time coordinate t and take the non-relativistic limit $|\dot{\vec{x}}| \ll c$ to determine the value of the constant α. Characterise the appearing terms.

b) Derive the equations of motion by varying the action in $(*)$. (You may set $c = 1$ from now on.) *Hint:* Calculate the canonically conjugate momentum P_μ first.

c) Show that the form of the action is invariant under reparametrisations $\tau' = f(\tau)$. This is what we call *manifestly* invariant.

d) Consider an electrically charged particle with charge q. In the presence of an external gauge field A_μ there is an additional term in the action governing the interaction between particle and field given by

$$S_{\text{em}} = \frac{q}{c} \int d\tau \, A_\mu(X) \, \frac{\partial X^\mu}{\partial \tau} \,.$$

Find the variation of $A^\mu(X)$ under a variation of the path δX^μ. Vary the action $S = S_{\text{rp}} + S_{\text{em}}$ w.r.t. X^μ to find the equations of motion for the particle. *Hint:* Use P_μ from above to simplify the expression.

3. Polynomial action (intermediate – hard)

There is another way to write the action of a relativistic particle. We introduce an auxiliary field called vierbein (or "einbein" in this case) e along the worldline of the particle and rewrite the action in the form

$$S_{\text{pp}} = \int d\tau (e^{-1}\dot{X}^2 - m^2 e).$$

a) Show that S_{pp} is equivalent to S_{rp} above by eliminating the einbein from the action.

b) Derive the equations of motion by varying S_{pp} with respect to X and e.

c) Show that S_{pp} is invariant under infinitesimal reparametrisations $\delta\tau = -\epsilon(\tau)$ to linear order in ϵ. First find the correct transformation of X^μ. The einbein transforms like (can you derive it?)

$$\delta e = \partial_\tau(\epsilon(\tau)e).$$

d) Reparametrisation invariance is a gauge invariance. Thus by fixing a gauge we can eliminate one degree of freedom. Assume a gauge in which e is constant. Show that e can be written like

$$e = \frac{\ell}{\tau_2 - \tau_1},$$

where ℓ is the invariant length of the worldline for a path starting at $X^\mu(\tau_1)$ and ending at $X^\mu(\tau_2)$. *Hint:* Meditate on the role of the einbein and on how to define ℓ.