Introduction to String Theory — Problem Set 2

ETH Zurich, HS11 — B. Schwab, Prof. N. Beisert

1. Symmetries of the classical string (intermediate)

In this exercise we examine the classical symmetries of the Polyakov string

$$S_P = -\frac{T}{2}\int d^2\xi\,\sqrt{-g}\,g^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta X^\nu\,\eta_{\mu\nu}.$$

We start with the global symmetries – Lorentz and translational symmetry – and proceed to gauge symmetries – reparametrisation and Weyl symmetry.

a) Consider the transformation

$$X^\mu \to \Lambda^\mu{}_\nu X^\mu + a^\mu$$

which is a combination of a Lorentz transformation and a translation, a.k.a. a Poincaré transformation. Using the Noether procedure show that in conformal gauge $g_{\alpha\beta} = \eta_{\alpha\beta}$ the Noether currents corresponding to these symmetries are given by

$$\mathcal{P}^\alpha_\mu = T\partial^\alpha X_\mu, \qquad \mathcal{J}^\alpha_{\mu\nu} = \mathcal{P}^\alpha_\mu X_\nu - \mathcal{P}^\alpha_\nu X_\mu.$$

b) Find and identify the conserved charges associated with Lorentz boosts and time translations. *Hint*: For the Lorentz boost assume $X^0 = t$.

c) Show that the Polyakov string action is invariant under a reparametrisation $\xi^\alpha \to \tilde{\xi}^\alpha(\xi)$.

d) Show that the Polyakov string is also invariant under Weyl transformations: local length-changing but angle preserving transformations of the metric $g_{\alpha\beta} \to e^{2\omega(\xi)} g_{\alpha\beta}$.

e) Consider an infinitesimal Weyl transformation

$$\delta g_{\alpha\beta} = 2\omega g_{\alpha\beta} \quad \text{and} \quad \delta X^\mu = 0$$

and show that Weyl symmetry implies the vanishing of the trace of the worldsheet energy-momentum tensor

$$T^\alpha{}_\alpha = 0.$$

Hint: The variation of the determinant is given by

$$\delta \det g = -\det g\, g_{\alpha\beta}\,\delta g^{\alpha\beta}.$$

$\longrightarrow$

2. Classical spinning strings (easy – intermediate)

The classical solution for the wave equation is given by

$$X^\mu(\sigma,\tau) = X^\mu_{\mathrm{L}}(\tau+\sigma) + X^\mu_{\mathrm{R}}(\tau-\sigma),$$

the constraints by

$$\dot{X}\cdot X' = 0 \qquad \text{and} \qquad \dot{X}^2 + X'^2 = 0.$$

It will be beneficial to work in static gauge $X^0(\sigma,\tau) = R\tau$ (τ being the worldsheet time).

a) Show that

$$\begin{aligned} X^0 &= R\tau \\ X^1 &= R\cos(\sigma)\cos(\tau) \\ X^2 &= R\cos(\sigma)\sin(\tau) \end{aligned}$$

can be written in the form of the general solution of the wave equation and that it fulfils the constraints. Calculate the energy $P^0 = E$ and the angular momentum J_{ij} of the solution.

b) Show that

$$\begin{aligned} X^0 &= R\tau \\ X^1 &= R\cos(\sigma)\cos(\tau) \\ X^2 &= R\cos(2\sigma)\sin(2\tau) \end{aligned}$$

can be written in the form of the general solution of the wave equation but does not fulfil the constraint equations.

c) (optional) Closed strings can develop cusps. These points σ_0 on the string are indicated by a singularity in the parametrisation

$$\frac{\partial \vec{X}}{\partial t}(\sigma_0, t) = 0.$$

Show that the string reaches the speed of light at a cusp. Moreover, show that cusps move perpendicular to the direction of the string.

d) (advanced) Explain why cusps form generically in $3+1$ dimensions but not so in higher dimensions.