1. Virasoro algebra (intermediate)

In this exercise we want to investigate in detail the Virasoro algebra as it appears up in light cone string theory. For simplicity we will only work with the left movers L_n^{L} as the right movers L_n^{R} commute with these and satisfy an identical algebra. The mode operators α_n^i (with $i = 1, \ldots, D-2$) satisfy the algebra (we drop the L/R superscript)

$$[\alpha_n^i, \alpha_m^j] = m\delta^{ij}\delta_{n+m},$$

and the *normal ordered* Virasoro generators are given by

$$L_n = \frac{1}{2}\sum_{p\geq 0} \alpha_{n-p}^i \alpha_p^i + \frac{1}{2}\sum_{p<0} \alpha_p^i \alpha_{n-p}^i.$$

An algebra $\mathfrak{g}$ is a Lie algebra if its product $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (called the Lie bracket) is antisymmetric $[a, b] = -[b, a]$ $\forall a, b \in \mathfrak{g}$ and satisfies the Jacobi identity

$$\big[a, [b, c]\big] + \big[b, [c, a]\big] + \big[c, [a, b]\big] = 0 \;\forall a, b, c \in \mathfrak{g}.$$

a) Show that the commutator of two Virasoro generators with $m + n \neq 0$ is given by

$$[L_m, L_n] = \frac{1}{2}\sum_p p\, \alpha_{m-p}^i \alpha_{p+n}^i + (m-p)\alpha_{n+m-p}^i \alpha_p^i$$

b) By relabelling the summands, rewrite the above result in the following form

$$[L_m, L_n] = (m-n)L_{n+m}.$$

Argue that the complete solution, including the terms $n = -m$ is given by

$$[L_m, L_n] = (m-n)L_{m+n} + C(m)\delta_{m+n,0}$$

where $C(m)$ is a real valued, odd ($C(-m) = -C(m)$) function. The last term is called the *central extension* of the Virasoro algebra. Determine $C(m)$ up to two constants by considering the Jacobi identity. The solution is given by $C(m) = \frac{1}{12}(D-2)(m^3 - m)$.

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2. Analytical continuation of the ζ-function (intermediate – hard)

In the lecture you have seen the peculiar result of the sum $\zeta(-1) = \sum_{n=1}^{\infty} n = -\frac{1}{12}$. In this exercise we will try to understand where the result comes from by analytically continuing the ζ-function. The Γ and ζ functions of a complex variable z are given by

$$\Gamma(z) = \int_0^\infty dt\, e^{-t} t^{z-1} \quad \text{and} \quad \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

a) We start by regularising $\zeta(-1)$ using a small parameter ϵ. Show that we can write the zeta function like $\zeta_\epsilon(-1) = -\frac{\partial}{\partial\epsilon} \sum_{n=1}^{\infty} e^{-n\epsilon}$ in the limit $\epsilon \to 0$. Argue that the sum in this expression is convergent and give the solution. Expand the expression for small ϵ and show that the result is given by $\zeta_\epsilon(-1) \approx \frac{1}{\epsilon^2} - \frac{1}{12} + \mathcal{O}(\epsilon)$.

b) Show that for $\mathrm{Re}(z) > 1$ you can write

$$\Gamma(z)\zeta(z) = \int_0^\infty \frac{dt\, t^{z-1}}{e^t - 1}\,.$$

Conclude that it is possible to rewrite the integral to give

$$\begin{aligned}\Gamma(z)\zeta(z) = &\int_0^1 dt\, t^{z-1} \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12} \right) + \frac{1}{z-1} - \frac{1}{2z} + \frac{1}{12(z+1)} \\ &+ \int_1^\infty \frac{dt\, t^{z-1}}{e^t - 1}\,.\end{aligned}$$

c) (advanced) The right hand side is well defined for $\mathrm{Re}(z) > -2$ (why?). We know that $\Gamma(z)$ has poles for $z = 0, -1, -2, \ldots$ with residues

$$\mathrm{Res}_{z_0=-n}\big[\Gamma(z_0)\big] = \frac{(-1)^n}{n!}\,.$$

Conclude that the values of $\zeta(z)$ at $z = 0$ and $z = -1$ are

$$\zeta(0) = -\frac{1}{2} \quad \text{and} \quad \zeta(-1) = -\frac{1}{12}\,.$$

3. Poincaré transformations (easy)

Poincaré transformations $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu + a^\mu$ form a group whose product is defined as $T(\Lambda_1, a_1)T(\Lambda_2, a_2) = T(\Lambda_1\Lambda_2, a_1 + \Lambda_1 a_2)$. The inverse reads $T(\Lambda, a)^{-1} = T(\Lambda^{-1}, -\Lambda^{-1}a)$. Consider an infinitesimal transformation with generators $\mathcal{J}$ and $\mathcal{P}$

$$T(1 + \omega, \epsilon) = 1 + \frac{i}{2}\omega^{\mu\nu}\mathcal{J}_{\mu\nu} - i\epsilon^\mu \mathcal{P}_\mu + \ldots.$$

They define the Lie algebra of the Poincaré group. Show that

$$T(\Lambda, a)\, T(1 + \omega, \epsilon)\, T(\Lambda, a)^{-1} = T\big(\Lambda(1 + \omega)\Lambda^{-1}, \Lambda\epsilon - \Lambda\omega\Lambda^{-1}a\big).$$

How do $\mathcal{J}$ and $\mathcal{P}$ transform under $T(\Lambda, a)$? What relations do you get when you take Λ, a to be infinitesimal as well?