Introduction to String Theory — Problem Set 6

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1. Kaluza-Klein theory (warm-up)

Already as early as 1921 T. Kaluza proposed a model to unify gravity and electromagnetism by introducing a (large) fifth dimension. O. Klein greatly improved this ansatz by introducing the idea of a compact fifth dimension. In Kaluza-Klein theory the metric tensor G_{MN} in 4+1 dimensions is split into 3 fields: a metric field $g_{\mu\nu}$ in 3+1 dimensions, a $U(1)$ gauge field A_μ and a scalar called the *dilaton* ϕ.

a) Begin by considering a massless scalar field Φ in D dimensions. Argue that compactifying one spacelike dimension on a circle of radius R:

$$x^{D-1} \sim x^{D-1} + 2\pi R$$

will give you an infinite tower of massive scalar fields in $D-1$ dimensions and one massless scalar. *Hint:* Contemplate the kind of momenta you get in compact spaces. Use the Fourier series. Write down the equations of motion of the scalar in D dimensions to see which $D-1$ dimensional modes are massive and which massless.

b) How can you make this tower of massive scalar fields vanish or at least impossible to detect at small energies?

c) (optional) Now we return to KK theory. The metric has the following form

$$G_{MN} = \phi^{-1/3} \begin{pmatrix} g_{\mu\nu} - \kappa^2 \phi A_\mu A_\nu & -\kappa\phi A_\mu \\ -\kappa\phi A_\nu & \phi \end{pmatrix}.$$

5D gravity is of course invariant under reparametrisations. However after singling out the fifth dimension we can only have 4D reparametrisations and a transformation

$$x^\mu = \tilde{x}^\mu \qquad \text{and} \qquad x^5 + f(x^\mu) = \tilde{x}^5.$$

Show that this transformation implies a gauge transformation on the field A_μ (which is contained in $G_{\mu 5}$).

We conclude by remarking that in KK theory the compactification of one dimension yields one massless metric field, one massless gauge field and one massless scalar along with three infinite towers of massive modes. Let us now turn to string theory.

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2. T-duality: Self-dual radius (intermediate)

When compactifying one dimension of string theory on a circle, new stringy effects appear that cannot be seen in field theory. One of these effects is *winding* of the string around the compactified dimension. We compactify the coordinate $D-2$ on a circle (drop the index $D-2$ for simplicity) and request that

$$X^{D-2}(\sigma + 2\pi, \tau) = X^{D-2}(\sigma, \tau) + 2\pi\kappa^2 w$$

where $w = mR/\kappa^2$ is the *winding*. The left and right movers are then defined as usual for all directions except for the compact dimension where

$$X_{\mathrm{L}} = \frac{1}{2}x + \frac{\kappa^2}{2}(p+w)\xi^{\mathrm{L}} + \frac{i\kappa}{\sqrt{2}}\sum_{n\neq 0}\frac{\alpha_n^{\mathrm{L}}}{n}\exp(-in\xi^{\mathrm{L}}),$$

$$X_{\mathrm{R}} = \frac{1}{2}x + \frac{\kappa^2}{2}(p-w)\xi^{\mathrm{R}} + \frac{i\kappa}{\sqrt{2}}\sum_{n\neq 0}\frac{\alpha_n^{\mathrm{R}}}{n}\exp(-in\xi^{\mathrm{R}}).$$

a) Derive the $(D-1)$-dimensional mass-squared $M^2 = -p^2$ of states in the presence of a compact dimension in terms of the level operators N^{L} and N^{R}, the winding number m and compact momentum $p = n/R$. *Hint:* Use the Virasoro constraints $L_0^{\mathrm{L}} = L_0^{\mathrm{R}} = a$.

b) Show that the level matching constraint $N^{\mathrm{L}} = N^{\mathrm{R}}$ does not hold for strings with both winding number m and Kaluza-Klein momentum number n not equal zero. What happens to these states at $R \to \infty$?

c) Consider the mass formula for the cases

$$m = n = 0; \quad m = 0, n \neq 0; \quad m \neq 0, n = 0; \quad m = n = \pm 1; \quad m = -n = \pm 1.$$

For which values of N^{L} and N^{R} does the spectrum (possibly) contain tachyonic and massless states? What is their spin (scalar, vector, tensor) as viewed from $D-1$ non-compact spacetime dimensions?

d) In the case of non-zero winding and non-zero compact momentum ($m = n = \pm 1$ and $m = -n = \pm 1$) show that there is a special radius R^* where some states become massless. What happens for the other cases at this radius?

Nota bene: What happened here? After compactifying one dimension we were left with two massless vector states with $U(1) \times U(1)$ gauge group, that is one more than in KK theory. Choosing the self-dual radius R^* produces four additional massless vectors which combine with the generic ones to $SU(2) \times SU(2)$ gauge fields. What we witness here is symmetry enhancement at the selfdual point! These are true string theory effects that cannot appear in KK theory.