1. Two-point function (easy)

In this exercise we want to compute the closed string propagator

$$\left\langle X^\mu(z,\bar{z}) X^\nu(z',\bar{z}')\right\rangle = -\frac{\kappa^2}{2}\eta^{\mu\nu}\log|z-z'|^2$$

which is given by the difference of the time-ordered and the normal ordered product of the operators $X^\mu(z,\bar{z})$ and $X^\nu(z',\bar{z}')$. Assume $|z| > |z'|$. *Hint*: You may ignore the effects of the centre of mass coordinates x^μ or use $:p^\mu x^\nu: = x^\nu p^\mu$.

2. Conformal transformations (intermediate)

Consider conformal transformations $z \to z'(z)$. Primary fields transform as tensors under conformal transformations

$$\mathcal{O}'(z,\bar{z}) = \left(\frac{\partial z'}{\partial z}\right)^h \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{\bar{h}} \mathcal{O}(z'(z),\bar{z}'(\bar{z}))$$

a) How does a primary field transform under infinitesimal transformations $z' \to z+\zeta(z)$?

b) Show that the operator $:e^{ikX}:$ for a single scalar field X is primary by computing its OPE with the stress-energy tensor. Determine the conformal weights h and $\bar{h}$.

3. The complex logarithm (intermediate)

The propagator that you calculated above and the two-point function

$$\left\langle \partial X^\mu(z,\bar{z}) \bar{\partial} X^\nu(z',\bar{z}')\right\rangle = \pi\kappa^2\eta^{\mu\nu}\delta^2(z-z',\bar{z}-\bar{z}')$$

are related by the two derivatives. Show that

$$\partial\bar{\partial}\log|z|^2 = 2\pi\delta^2(z,\bar{z})$$

a) ... by considering the divergence theorem.

b) ... by regulating the singularity at $z=0$.

$\longrightarrow$

4. Free fermions (intermediate)

In QFT, fermions are represented by *Grassmann-valued* fields. In contradistinction to ordinary numbers, Grassmann numbers η_1, η_2 *anticommute* with each other

$$\eta_1 \eta_2 = -\eta_2 \eta_1.$$

For instance, the Pauli exclusion principle is realised by the anticommuting Grassmann numbers. We introduce a free real fermion field in two dimensions with the action

$$S = \frac{1}{2\pi} \int d^2 z \; \psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi}.$$

The field has the OPE

$$\psi(z)\psi(w) = -\psi(w)\psi(z) = \frac{1}{z-w} + \dots$$

and similar for $\bar{\psi}$. The strange looking minus sign is due to the anti-commutative nature of Grassmann numbers. The stress-energy tensor for the free fermion reads

$$T(z) = -\frac{1}{2} {:}\psi \partial \psi{:}.$$

a) Show that ψ is a primary field with weight $(\frac{1}{2}, 0)$.

b) Compute the OPE of two T's and give the central charge of the theory.