1. Veneziano Amplitude (intermediate – hard)

The Veneziano amplitude[1] led to the discovery of string theory. In this problem we will attempt to calculate this amplitude. The open string tachyon vertex operator is given by an integral over the boundary of the string

$$V(k) = \sqrt{g_s} \int dx \, {:}\exp(ik_\mu X^\mu(x)){:}$$

such that the four tachyon scattering amplitude $A_4(k_1, k_2, k_3, k_4)$ is given by the expression

$$A_4(k_1, k_2, k_3, k_4) \sim \frac{1}{g_\text{s}} \big\langle V_1 \ldots V_4 \big\rangle \sim g_\text{s} \int\limits_{x_i < x_{i+1}} \prod_{i=1}^{4} dx_i \big\langle {:}e^{ik_1 \cdot X(x_1)}{:} \ldots {:}e^{ik_4 \cdot X(x_4)}{:} \big\rangle.$$

The ordering of insertions x_i is due to Chan–Paton factors.

a) The expectation value is computed using Wick's theorem and the two-point correlator

$$\big\langle X^\mu(x) X^\nu(y) \big\rangle = -2\kappa^2 \eta^{\mu\nu} \log|x-y|.$$

A generic Wick contraction will have n_{ij} correlators between the pair of points (x_i, x_j). Count the combinatorial number of contractions that lead to a configuration specified by the numbers n_{ij}. *Hint:* You need to split $n_i = \sum_j n_{ij}$ fields at point x_i into n_{ij} contractions to point x_j.

b) Collect the combinatorial factors and expansion coefficients of $e^{ik_i \cdot X(x_i)}$. Then perform the sum over n_{ij}'s. *Hint:* You should obtain the following integral expression for A_4 (up to the momentum-conserving delta function which is more subtle)

$$A_4 \sim g_s \delta^{26}\left(\sum_i k_i\right) \int \prod_{i=1}^{4} dx_i \prod_{j<l} |x_j - x_l|^{2\kappa^2 k_j \cdot k_l}.$$

c) Show that the integral is invariant under the $SL(2,\mathbb{R})$ Möbius transformation

$$x_i \to \frac{ax_i + b}{cx_i + d}$$

for on-shell momenta $k_i^2 = \kappa^{-2}$. *Hint:* Use momentum conservation $\sum_i k_i = 0$.

d) The integral given above is divergent because it has the non-compact Möbius group as a symmetry. It thus contains an irrelevant factor of the group volume which is infinite. We divide by the latter and use the symmetry to set $x_1 = 0$, $x_2 = x$, $x_3 = 1$ and $x_4 \to \infty$. Explain why the amplitude after the transformation reduces to

$$A_4 \sim g_s \delta^{26}\left(\sum_i k_i\right) \int dx \; |x|^{2\kappa^2 k_1 \cdot k_2} |1-x|^{2\kappa^2 k_2 \cdot k_3} + (k_2 \leftrightarrow k_3).$$

What is the integration range of x_2 now? Why? What happened to the normalisation in front of the integral?

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[1] For the original paper see http://dx.doi.org/10.1007/BF02824451

e) The resulting integral is well known. It is in the form of the Euler beta function

$$B(a,b) = \int_0^1 dy\; y^{a-1}(1-y)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Write down the solution for the amplitude. *Hint:* Use the Mandelstam variables

$$s = -(k_1+k_2)^2, \quad t = -(k_1+k_3)^2, \quad u = -(k_1+k_4)^2$$

to simplify the result.

f) Where does this amplitude have poles? What do these poles correspond to?

2. Schwarzian derivative (intermediate)

The stress-energy tensor transforms under a finite conformal transformation $z \to z' = f(z)$ as

$$T(z) \to T'(z) = (\partial f)^2 T(z') + \frac{c}{12}\mathcal{S}(z', \bar{z}')$$

where

$$\mathcal{S}(z', \bar{z}') = \frac{\partial f(z)\partial^3 f(z) - \frac{3}{2}(\partial^2 f(z))^2}{(\partial f)^2}$$

is the Schwarzian derivative.

a) Show that the Schwarzian derivative reproduces the correct infinitesimal transformation.

b) Show that the Schwarzian derivative has the correct property under successive conformal transformations.

c) Prove that for a transformation

$$f(z) = \frac{az+b}{cz+d}$$

the Schwarzian derivative yields $\mathcal{S}(z, \bar{z}) = 0$. Why is this not surprising?