Statistical Physics
Exercise Sheet 12

Exercise 12.1 The mean-field quantum Heisenberg model

We consider the Heisenberg model for a ferromagnet on a hypercubic lattice of d dimensions with *quantum* spins described by operators $\vec{S}_i$:

$$H = -J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j.$$

The spin size is S, meaning that the spins can take on the values $-S, -S+1, \ldots, S-1, S$ along their quantization axis. Discuss this model within the mean field approximation (MFA).

a) Construct the mean-field Hamiltonian for the quantum Heisenberg model, by writing:

$$\vec{S}_i = \langle S_i \rangle + \delta\vec{S}_i = \vec{m} + (\vec{S}_i - \vec{m}),$$

where $\vec{m}$ is the magnetization per site. Follow the scheme discussed in the lecture by neglecting terms quadratic in $\delta\vec{S}_i$. Interpret the resulting mean-field Hamiltonian.

b) Compute the partition function and compare the result for $S = 1/2$ with the result for the Ising model in the lecture notes.

Hint: It is convenient to take the quantization axis parallel to $\vec{m}$. When computing the partition function you encounter a finite geometric series evaluated by

$$\sum_{x=0}^{n} r^x = \frac{1 - r^{n+1}}{1 - r}.$$

c) Compute the free energy and minimize it with respect to the magnetization $m = |\vec{m}|$. Show that the resulting (self-consistency) equation can be written as

$$\frac{m}{S} = \mathrm{B}_S(\beta J z m S),$$

where $\mathrm{B}_S(x)$ is called the Brillouin function, defined by

$$\mathrm{B}_S(x) = \frac{2S+1}{2S} \coth\left(\frac{2S+1}{2S}x\right) - \frac{1}{2S}\coth\left(\frac{1}{2S}x\right).$$

Also obtain an expression for the critical temperature T_C.

Hint: Use the expansion $\coth(x) = x^{-1} + x/3 - x^3/45 + \cdots$.

d) Next, assume that the spins in the above model are not quantum spins, but classical spins $\vec{S}_i$, redo the MFA and compare the results for $S \to \infty$ and $S = 1/2$ to the quantum case.

e) Find the behaviour of the magnetization near T_C. In particular, extract the critical exponent β in

$$m(T) \propto |T_C - T|^{\beta}.$$

Hint: Use the above expansion for B_S in the self-consistency equation.

Office Hours: Monday, December 10th, 08:00-10:00, HIT K 23.7 (Evert van Nieuwenburg)