Sheet IV

Due: week of October 22

Question 1 [*Gauss curvature versus Riemannian sectional curvature*]: Consider a Riemannian manifold M of dimension $n > 2$. For a point $p \in M$ let $\Pi \subset T_pM$ be a two-plane. Let G_Π be the geodesic plane at p with respect to Π, i.e.

$$G_\Pi = \exp(\Pi) = \{\exp(X) : X \in \Pi\}. \tag{1}$$

Show that the Gauss curvature of the surface G_Π at p is equal to the Riemannian sectional curvature $R(E_1, E_2, E_1, E_2)$, where $\{E_1, E_2\}$ is an orthonormal basis for Π.

Hint: Set up Riemannian normal coordinates $(x^1, \ldots, x^n)$ with origin at p. In these coordinates the metric is given by

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3} R_{ikjl} x^k x^l + O(r^3), \tag{2}$$

where $r = \sqrt{\sum_{i=1}^n (x^i)^2}$ (this expansion is part of the result of question 2 of sheet III). Set up the coordinates such that G_Π is given in these coordinates by

$$G_\Pi = \{p \in M : x^3 = \ldots = x^n = 0\}, \tag{3}$$

i.e. E_1 and E_2 span Π. Introduce polar coordinates r, ϕ in G_Π and calculate the circumference C_r of a circle $S_r : \sqrt{(x^1)^2 + (x^2)^2} = r$ using

$$C_r = \int_0^{2\pi} \sqrt{g_{ij} \frac{dx^i}{d\phi} \frac{dx^j}{d\phi}} d\phi. \tag{4}$$

Here g_{ij} is the induced metric on G_Π. Recall from the lecture that the Gauss curvature K at the point p is given by

$$\frac{dC_r}{dr} = 2\pi - \pi K r^2 + O(r^3). \tag{5}$$

Question 2 [*Riemannian spaces of constant curvature*]: On a domain of $\mathbb{R}^n$ consider the conformally flat metric

$$g_{ij} = \Omega^2 \delta_{ij} \tag{6}$$

and set

$$\Omega = \frac{1}{1 + \frac{c}{4} r^2}, \tag{7}$$

where $r = \sqrt{\sum_{i=1}^n (x^i)^2}$ and c is a constant.

(i) Show that every plane Π has sectional curvature K_Π equal to c.

Hint: Show that

$$\Gamma^k_{ij} = \Omega^{-1}(\delta^k_i \partial_j \Omega + \delta^k_j \partial_i \Omega - \delta_{ij}\partial_k \Omega) \tag{8}$$

and show that R_{ijkl} reduces to

$$R_{ijkl} = c(g_{ik}g_{jl} - g_{il}g_{jk}). \tag{9}$$

(ii) Consider the case $c = 1$. Show that $(\mathbb{R}^n, g)$ is incomplete.

Hint: Show that the distance between the origin and a point at infinity is finite.

(iii) Consider the case $c = -1$. Show that points at the distance $r = 2$ from the origin are at infinity.

Remarks: To (ii): Adding the point at infinity we get the sphere S^n. To (iii): This space is the hyperbolic space $\mathbb{H}^n$.

Question 3 [*Lorentzian spaces of constant curvature*]: On a domain of $\mathbb{R}^{n+1}$ consider the conformally flat metric

$$g_{\mu\nu} = \Omega^2 \eta_{\mu\nu} \tag{10}$$

and set

$$\Omega = \frac{1}{1 + \frac{c}{4}\sigma}, \tag{11}$$

where $\sigma = \eta_{\mu\nu}x^\mu x^\nu = -(x^0)^2 + r^2$, $r = \sqrt{\sum_{i=1}^n (x^i)^2}$ and c is a constant.

(i) Show that every spacelike plane has sectional curvature equal to c.

Hint: Show that

$$\Gamma^\mu_{\alpha\beta} = \Omega^{-1}(\delta^\mu_\alpha \partial_\beta \Omega + \delta^\mu_\beta \partial_\alpha \Omega - \eta_{\alpha\beta}\eta^{\mu\nu}\partial_\nu \Omega) \tag{12}$$

and show that $R_{\mu\nu\alpha\beta}$ reduces to

$$R_{\mu\nu\alpha\beta} = c(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}). \tag{13}$$

(ii) Show that every timelike plane has sectional curvature equal to $-c$.

(iii) Consider the case $c = 1$. Find the domain in $\mathbb{R}^{n+1}$ where the metric is defined. Show that the x^0-axis is a geodesic and show that the proper time (i.e. the arc length along the geodesic measured from the origin) becomes infinite at the point $(2, 0, \dots, 0)$.

(iv) Consider the case $c = -1$. Find the domain in $\mathbb{R}^{n+1}$ where the metric is defined. Show that the x^0-axis is a geodesic. Show that the past of the point with coordinates $(2, 0, \dots, 0)$ is the whole hyperplane $x^0 = 0$. Consider the points p' with coordinates $(2 + \varepsilon, 0, \dots, 0)$ and p'' with coordinates $(-2 - \varepsilon, 0, \dots, 0)$. Show that p', p'' can be connected by a curve with arbitrarily large arc length (argue without setting up an integral for the arc length).

Remark: The space in (iii) is called de Sitter space the one in (iv) anti de Sitter.