Sheet IX

Due: week of December 10

The present exercise sheet deals with decay properties of the solution of the scalar wave equation

$$\Box\phi = -4\pi\rho \tag{1}$$

with trivial initial data, i.e.

$$\phi|_{t=t_0} = \left.\frac{\partial\phi}{\partial t}\right|_{t=t_0} = 0 \tag{2}$$

in the limit $t_0 \to -\infty$. We assume ρ to have compact spatial support. The solution is given by the full retarded integral

$$\phi(t,x) = \int_{C^-(t,x)} \frac{\rho(t',x')}{|x-x'|} d^3x', \tag{3}$$

where $C^-(t,x)$ denotes the past light cone with vertex at (t,x). In the following, a limit of the form

$$\lim_{(u,\xi);\, r\to\infty} \tag{4}$$

will mean taking the limit as $r\to\infty$ along a generator of the future light cone C_u^+, the generator being given by $\xi \in S^2$. We define

$$\Phi(u,\xi) := \lim_{(u,\xi);\, r\to\infty} r\phi. \tag{5}$$

Recall from the lecture that

$$\Phi(u,\xi) = \int_{P(u,\xi)} \rho = \int_{\mathbb{R}^3} \rho(u+\xi\cdot x', x') d^3x', \tag{6}$$

where $P(u,\xi)$ is the null hyperplane given by

$$P(u,\xi) := \{(t',x') : t' = u+\xi\cdot x', x' \in \mathbb{R}^3\}. \tag{7}$$

Question 1 [*Tangential decay*]: Show that

$$\lim_{(u,\xi);\, r\to\infty} r\Omega_i\phi = \Omega_i\Phi, \tag{8}$$

where Ω_i $(i=1,2,3)$ are the rotation fields given by

$$\Omega_i := \varepsilon_{ijk} x^j \frac{\partial}{\partial x^k}. \tag{9}$$

Hints:

(i) Use polar coordinates (θ, φ) on S^2 such that

$$\xi^1 = \sin\theta\cos\varphi, \qquad \xi^2 = \sin\theta\sin\varphi, \qquad \xi^3 = \cos\theta \tag{10}$$

and arrange the coordinate axes in such a way that

$$\Omega_3 = \frac{\partial}{\partial\varphi} = x^1\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^1}. \tag{11}$$

In this setting it suffices to show that

$$\lim_{(u,\xi);\, r\to\infty} r\Omega_3\phi = \frac{\partial\Phi}{\partial\varphi}. \tag{12}$$

(ii) To show (12) make use of the commutation relation $[\Omega_i, \Box] = 0$ together with (1), (3), (5) and (6) with $\Omega_3\phi$ in the role of ϕ to show that

$$\lim_{(u,\xi);\, r\to\infty} r\Omega_3\phi = \int_{P(u,\xi)} \Omega_3\rho. \tag{13}$$

Question 2 [*Longitudinal decay*]: Show that

$$\lim_{(u,\xi);\, r\to\infty} r^2 L\phi = -\Phi. \tag{14}$$

Hints:

(i) Compute the commutator $[S, \Box]$, where S is the scaling field

$$S := x^\mu\frac{\partial}{\partial x^\mu}. \tag{15}$$

Use the result together with (1), (3), (5) and (6) with $S\phi$ in the role of ϕ to show that

$$\lim_{(u,\xi);\, r\to\infty} rS\phi = \int_{P(u,\xi)} (S\rho + 2\rho). \tag{16}$$

(ii) Show that

$$x'^i\frac{\partial}{\partial x'^i}\rho(u+\xi\cdot x', x') = \left(t\frac{\partial\rho}{\partial t} + x^i\frac{\partial\rho}{\partial x^i}\right)(u+\xi\cdot x', x') - u\frac{\partial\rho}{\partial t}(u+\xi\cdot x', x'). \tag{17}$$

Integrate (17) on $\mathbb{R}^3$ to show that (16) is equal to

$$-\Phi + u\frac{\partial\Phi}{\partial u}. \tag{18}$$

(iii) Show that

$$S = \frac{1}{2}(u\underline{L} + vL) \tag{19}$$

and use (see the lecture)

$$\lim_{(u,\xi);\, r\to\infty} rL\phi = 0, \qquad \lim_{(u,\xi);\, r\to\infty} r\underline{L}\phi = 2\frac{\partial\Phi}{\partial u} \tag{20}$$

together with $u = t - r$, $v = t + r$ to show (14).