4 Symmetries

So far we have not discussed symmetries. QFT does not actually need symmetries, but they help very much in restricting classes of models, providing stability and simplifying calculations as well as results.[1]

For example, in most cases QFT's have some symmetry of space and time. Particularly in fundamental particle physics all models have relativistic invariance or Poincaré symmetry.

Symmetries are some transformations of the fields $\phi \to \phi'$ that map solutions of the equations of motion to other solutions. Hence they can be used to generate a whole class of solutions from a single one.

We shall discuss the action of various types of symmetries, their groups and representations, and the resulting conserved charges via Noether's theorem. Most of the discussion applies to classical and quantum field theories.

4.1 Internal Symmetries

Let us first discuss internal symmetries. In a QFT with several fields, these typically transform the fields into each other in some way without making reference to their dependence on space or time.

The simplest example is a complex scalar field $\phi(x)$ with Lagrangian and corresponding equation of motion

$$\mathcal{L} = -\partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi, \qquad \begin{aligned} \partial^2\phi - m^2\phi &= 0, \\ \partial^2\phi^* - m^2\phi^* &= 0. \end{aligned} \tag{4.1}$$

Consider a global transformation of the fields

$$\phi'(x) = e^{+i\alpha}\phi(x), \qquad \phi^{*\prime}(x) = e^{-i\alpha}\phi^*(x). \tag{4.2}$$

It maps a solution of the equations of motion to another solution[2]

$$\partial^2\phi' - m^2\phi' = e^{i\alpha}\left(\partial^2\phi - m^2\phi\right) = 0. \tag{4.3}$$

Moreover the symmetry leaves the Lagrangian and the action invariant

$$\mathcal{L}(\phi', \partial_\mu\phi') = \mathcal{L}(\phi, \partial_\mu\phi), \qquad S[\phi'] = S[\phi]. \tag{4.4}$$

[1] For free particles symmetries are not that helpful, the true power of symmetries arises in interacting situations.

[2] The transformed field ϕ' satisfies the equation of motion because ϕ does. Same for complex conjugate field ϕ^*.

Any such transformation must be a symmetry because it maps extrema of the action to extrema and hence solutions to solutions. Symmetries of the action are more powerful than mere symmetries of the equations of motion. In the following we will only consider symmetries of the action.[3]

Noether's Theorem. Every continuous global symmetry of the action leads to a conserved current and thus a conserved charge for solutions of the equations of motion.

Let us derive the theorem: Consider a solution ϕ of the equations of motion. By construction, any variation of the Lagrangian is a total derivative[4]

$$\begin{aligned}\delta\mathcal{L} &= \frac{\delta\mathcal{L}}{\delta\phi}\,\delta\phi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\,\partial_\mu\delta\phi \\ &= \partial_\mu\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\,\delta\phi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\,\partial_\mu\delta\phi = \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\,\delta\phi\right). \end{aligned} \tag{4.5}$$

Suppose now $\delta\phi$ is the infinitesimal field variation of a continuous symmetry. We know that $\delta S = 0$, hence the Lagrangian can only change by some total derivative

$$\delta\mathcal{L} = \delta\alpha\,\partial_\mu J_0^\mu. \tag{4.6}$$

Equating the two expressions for $\delta\mathcal{L}$ we find a current[5]

$$J^\mu = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\,\frac{\delta\phi}{\delta\alpha} - J_0^\mu \tag{4.7}$$

which is conserved for every solution ϕ

$$\partial_\mu J^\mu = 0. \tag{4.8}$$

Furthermore, a conserved current implies a conserved charge

$$Q(t) = \int d^d\vec{x}\,J^0(t,\vec{x}) \tag{4.9}$$

if we assume that the field vanishes sufficiently fast at spatial infinity

$$\dot{Q} = \int d^d\vec{x}\,\partial_0 J^0 = -\int d^d\vec{x}\,\partial_k J^k = 0. \tag{4.10}$$

The conserved charge actually generates an infinitesimal symmetry transformation via the Poisson brackets

$$[Q,F] = -\frac{\delta F}{\delta\alpha} \tag{4.11}$$

as can be shown using its defining relations.

[3] For example the scaling transformation $\phi(x) \to e^\beta\phi(x)$ also maps solutions to solutions, but it rescales the Lagrangian $\mathcal{L}' = e^{2\beta}\mathcal{L}$. If one considers QFT's to be specified by their Lagrangians, then this symmetry of the equations of motion relates two different models $\mathcal{L}$ and $\mathcal{L}'$. We typically us the freedom to redefine the fields to bring the Lagrangian in some canonical form.

[4] Usually we can ignore this term, here it is relevant.

[5] Any term of the form $\partial_\nu B^{\mu\nu}$ with antisymmetric indices on $B^{\mu\nu}$ can be added to J^μ without modifying any of the following relations.

Example. Let us consider the complex scalar field. The field variation is defined by

$$\delta\phi = i\phi\,\delta\alpha, \qquad \delta\phi^* = -i\phi^*\,\delta\alpha. \tag{4.12}$$

The Lagrangian is invariant under the transformation $\delta\mathcal{L} = 0$, hence $J_0^\mu = 0$. The other term reads

$$\begin{aligned} J^\mu &= \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\,\frac{\delta\phi}{\delta\alpha} + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^*)}\,\frac{\delta\phi^*}{\delta\alpha} \\ &= (-\partial^\mu\phi^*)\,(i\phi) + (-\partial^\mu\phi)\,(-i\phi^*) \\ &= -i(\partial^\mu\phi^*\phi - \phi^*\partial^\mu\phi). \end{aligned} \tag{4.13}$$

The naive divergence of the current reads

$$\partial_\mu J^\mu = -i(\partial^2\phi^*\phi - \phi^*\partial^2\phi). \tag{4.14}$$

This indeed vanishes for a solution of the equations of motion.

The conserved charge reads

$$Q = i\int d^d\vec{x}\,(\dot\phi^*\phi - \phi^*\dot\phi) = i\int d^d\vec{x}\,(\pi\phi - \phi^*\pi^*). \tag{4.15}$$

Transformed to momentum space we get

$$Q = \int \frac{d^d\vec{p}}{(2\pi)^d\,2e(\vec{p})}\,\big(a^*(\vec{p})a(\vec{p}) - b^*(\vec{p})b(\vec{p})\big). \tag{4.16}$$

This charge is indeed time-independent and (Poisson) commutes with the Hamiltonian. As expected, it obeys

$$\{Q,\phi\} = -i\phi = -\frac{\delta\phi}{\delta\alpha}\,, \qquad \{Q,\phi^*\} = +i\phi^* = -\frac{\delta\phi^*}{\delta\alpha}\,. \tag{4.17}$$

We furthermore observe a relation to the number operators

$$Q = N_a - N_b. \tag{4.18}$$

In the quantum theory, Q therefore measures the number of particles created by $a^\dagger$ minus the number of particles created by $b^\dagger$.

Despite the similarities, there is a crucial difference to the number operator: The charge Q is associated to a symmetry, whereas a single number operator N is not.[6] In a symmetric theory with interactions, Q is conserved while N is in general not.

Quantum Action. Let us briefly state how to represent this symmetry in the quantum theory where $Q = N_a - N_b$ becomes a quantum operator. It is obviously hermitian

$$Q^\dagger = Q. \tag{4.19}$$

[6] One might construct a non-local symmetry transformation corresponding the number operator in a free field theory. However, this symmetry would not generalise to interactions.

It obeys the following commutation relations with creation and annihilation operators

$$[Q, a(x)] = -a(x), \qquad [Q, b(x)] = +b(x),$$
$$[Q, a^\dagger(x)] = +a^\dagger(x), \qquad [Q, b^\dagger(x)] = -b^\dagger(x). \tag{4.20}$$

This tells us that particles of type a carry positive unit charge while the antiparticles of type b carry negative unit charge.

The commutators of spacetime fields $\phi \sim a^\dagger + b$ read

$$[Q, \phi(x)] = +\phi(x), \qquad [Q, \phi^\dagger(x)] = -\phi^\dagger(x), \tag{4.21}$$

which tell us that ϕ and $\phi^\dagger$ carry charges $+1$ and -1, respectively. The commutators are also in agreement with the classical result that charges generate infinitesimal transformations, i.e.

$$[Q, \phi] = +\phi = -i\,\frac{\delta\phi}{\delta\alpha}, \qquad [Q, \phi^\dagger] = -\phi^\dagger = -i\,\frac{\delta\phi^\dagger}{\delta\alpha}. \tag{4.22}$$

For finite transformations we introduce the operator

$$U(\alpha) = \exp(i\alpha Q). \tag{4.23}$$

We can convince ourselves that it obeys the following algebra with the fields[7]

$$U(\alpha)\,\phi(x)\,U(\alpha)^{-1} = e^{+i\alpha}\phi(x) = \phi'(x),$$
$$U(\alpha)\,\phi^\dagger(x)\,U(\alpha)^{-1} = e^{-i\alpha}\phi^\dagger(x) = \phi^\dagger{}'(x). \tag{4.24}$$

So $U(\alpha)$ generates a finite symmetry transformation by means of conjugation while Q generates the corresponding infinitesimal transformation by means of commutators.

Note that the operator $U(\alpha)$ is unitary because Q is hermitian

$$U(\alpha)^\dagger = \exp(-i\alpha Q^\dagger) = \exp(-i\alpha Q) = U(-\alpha) = U(\alpha)^{-1}. \tag{4.25}$$

A crucial property of symmetries in QFT is that they are represented by *unitary* operators. This is required to make expectation values invariant under symmetry.

The symmetry group for the complex scalar is simply U(1).

The above discussions only applied to operators, let us finally discuss transformations for states. States transform under finite transformations as

$$|\Psi'\rangle = U(\alpha)|\Psi\rangle. \tag{4.26}$$

Typically the vacuum is uncharged under symmetries[8]

$$Q|0\rangle = 0. \tag{4.27}$$

The transformation for all other states in the Fock space then follows from the transformation of creation operators.

[7] Note that $Q\phi = \phi(Q+1)$ implies $\exp(i\alpha Q)\phi = \phi\exp(i\alpha(Q+1)) = e^{i\alpha}\phi\exp(i\alpha Q)$.

[8] This is not a requirement. In fact, a charged vacuum is related to spontaneous symmetry breaking and Goldstone particles, see QFT II. Note that ordering ambiguities arise in the determination of the charges, and are resolved by specifying the intended charge of the vacuum. Ordering ambiguities do not matter for commutators.

4.2 Spacetime Symmetries

Next we shall consider symmetries related to space and time. In relativistic theories these are the spatial rotations and Lorentz boosts (altogether called Lorentz symmetries) as well as spatial and temporal translations. In total they form the Poincaré group.

Translations. Let us start with simple translations in space and time

$$(x')^\mu = x^\mu - a^\mu. \tag{4.28}$$

We demand that the fields merely change by shifting the position argument

$$\phi'(x') = \phi(x). \tag{4.29}$$

In other words the new field evaluated at the new position equals the old field at the old position.[9] Explicitly,

$$\phi'(x) = \phi(x+a) \quad \text{or} \quad \delta\phi(x) = \delta a^\mu \partial_\mu \phi(x). \tag{4.30}$$

In order for translations to be a symmetry, we have to require that the Lagrangian does not explicitly depend on the position

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = 0, \qquad \mathcal{L}(\phi(x), \partial_\mu\phi(x), x) = \mathcal{L}(\phi(x), \partial_\mu\phi(x)). \tag{4.31}$$

Energy and Momentum. The Noether theorem equally applies to this situation, let us derive the associated currents and charges. The variation of the Lagrangian reads[10]

$$\delta\mathcal{L} = \delta a^\mu \left(\frac{\delta \mathcal{L}}{\delta\phi}\, \partial_\mu\phi + \frac{\delta\mathcal{L}}{\delta\partial_\nu\phi}\, \partial_\mu\partial_\nu\phi \right) = \delta a^\mu \partial_\mu(\mathcal{L}(\phi, \partial\phi)). \tag{4.32}$$

We therefore obtain a contribution $(J_0)^\mu_\nu = \delta^\mu_\nu \mathcal{L}$. All in all we obtain a vector of conserved currents $T^\mu{}_\nu$ (with $\partial_\mu T^\mu{}_\nu = 0$) where the index ν labels the $d+1$ dimensions for shifting

$$T^\mu{}_\nu = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\, \partial_\nu\phi - \delta^\mu_\nu \mathcal{L}. \tag{4.33}$$

This object is called energy-momentum (or stress-energy) tensor. For a real scalar it reads

$$T^\mu{}_\nu = -\partial^\mu\phi\, \partial_\nu\phi + \tfrac{1}{2}\delta^\mu_\nu\big((\partial\phi)^2 + m^2\phi^2\big). \tag{4.34}$$

The corresponding conserved charge is the momentum vector

$$P_\mu = \int d^d\vec{x}\, T^0{}_\mu = \int d^d\vec{x}\, \Big(\dot\phi\, \partial_\mu\phi + \tfrac{1}{2}\delta^0_\mu\big((\partial\phi)^2 + m^2\phi^2\big)\Big) \tag{4.35}$$

[9] In other words, the transformation is active. One could also define $\phi'(x) = \phi(x')$ corresponding to a passive transformation.

[10] The derivative in the last term is meant to act on the x-dependence within the arguments $\phi(x)$ and $\partial\phi(x)$ of $\mathcal{L}$.

We recover the Hamiltonian as its time component

$$H = P_0 = \int d^d\vec{x}\,\bigl(\tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}(\vec{\partial}\phi)^2 + \tfrac{1}{2}m^2\phi^2\bigr), \tag{4.36}$$

while the total spatial momentum simply reads

$$\vec{P} = \int d^d\vec{x}\,\dot{\phi}\,\vec{\partial}\phi. \tag{4.37}$$

Quantum Action. We have already encountered the quantum operators for energy and momentum. Recall that in momentum space they read

$$P_\mu = \int \frac{d^d\vec{p}}{(2\pi)^d\,2e(\vec{p})}\,p_\mu a^\dagger(\vec{p})a(\vec{p}). \tag{4.38}$$

Performing a quantum commutator with the field yields

$$[P_\mu, \phi(x)] = -i\partial_\mu\phi(x). \tag{4.39}$$

As before, we can introduce an operator $U(a)$ for finite shift transformations as the exponential

$$U(a) = \exp(ia^\mu P_\mu). \tag{4.40}$$

Conjugating a field with it yields the shifted field[11]

$$U(a)\phi(x)U(a)^{-1} = \exp(a^\mu\partial_\mu)\phi(x) = \phi(x+a) = \phi'(x). \tag{4.41}$$

Note that the operator $U(a)$ is unitary because P_μ is hermitian.

Lorentz Transformations. Next, consider Lorentz transformations

$$(x')^\mu = (\Lambda^{-1})^\mu{}_\nu x^\nu. \tag{4.42}$$

All upper (contravariant) indices transform according to the same rule as x^μ under Lorentz transformations, whereas lower (covariant) indices transform with the matrix Λ, just as ∂_μ does, e.g.

$$(\partial')_\mu = \Lambda^\nu{}_\mu\partial_\nu. \tag{4.43}$$

A product between a covariant and contravariant index is Lorentz invariant

$$(x')^\mu(\partial')_\mu = \Lambda^\rho{}_\mu(\Lambda^{-1})^\mu{}_\nu x^\nu\partial_\rho = x^\nu\partial_\nu. \tag{4.44}$$

The matrix Λ has the defining property that it leaves the metric $\eta_{\mu\nu}$ invariant

$$\eta'_{\mu\nu} = \eta_{\rho\sigma}\Lambda^\rho{}_\mu\Lambda^\sigma{}_\nu = \eta_{\mu\nu}. \tag{4.45}$$

[11] The exponentiated derivative $\exp(a^\mu\partial_\mu)\phi(x)$ generates all the terms in the Taylor expansion of $\phi(x+a)$ for small a.

We can write this relation also as

$$(\Lambda^{-1})^\mu{}_\nu = \eta_{\nu\sigma}\Lambda^\sigma{}_\rho\eta^{\rho\mu} =: \Lambda_\nu{}^\mu. \tag{4.46}$$

It implies that it makes no difference whether indices are raised or lowered before or after a Lorentz transformation. Correspondingly, scalar products between equal types of vectors are invariant.

Lorentz transformations combine spatial rotations (the matrix acts on two of the spatial dimensions)

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = \exp\begin{pmatrix} 0 & -\varphi \\ \varphi & 0 \end{pmatrix} \tag{4.47}$$

and Lorentz boosts (the matrix acts on time and one of the spatial dimensions)

$$\begin{pmatrix} \cosh\vartheta & \sinh\vartheta \\ \sinh\vartheta & \cosh\vartheta \end{pmatrix} = \exp\begin{pmatrix} 0 & \vartheta \\ \vartheta & 0 \end{pmatrix}. \tag{4.48}$$

There are also some discrete transformations which we shall discuss below. Here we restrict to proper orthochronous Lorentz transformations which form the Lie group $\mathrm{SO}^+(d,1)$.

We note that spatial rotations are generated by anti-symmetric matrices while Lorentz boosts are generated by symmetric matrices. Composing various such transformations in 2-dimensional subspaces of spacetime we conclude that Lorentz rotations are generated as

$$\Lambda^\mu{}_\nu = \exp(\omega)^\mu{}_\nu \tag{4.49}$$

where $\omega^\mu{}_\nu$ is a matrix satisfying

$$\omega^k{}_l = -\omega^l{}_k, \qquad \omega^0{}_k = \omega^k{}_0, \qquad \omega^0{}_0 = \omega^k{}_k = 0. \tag{4.50}$$

Lowering the first index $\omega_{\mu\nu} = \eta_{\mu\rho}\omega^\rho{}_\nu$, this is equivalent to an anti-symmetric matrix

$$\omega_{\mu\nu} = -\omega_{\nu\mu}. \tag{4.51}$$

Angular Momentum. For a (scalar) field the transformation reads

$$\phi'(x) = \phi(\Lambda x), \qquad \delta\phi = \delta\omega^\mu{}_\nu x^\nu \partial_\mu\phi. \tag{4.52}$$

Lorentz invariance of the action requires the Lagrangian to transform in the same way[12]

$$\delta\mathcal{L} = \delta\omega^\mu{}_\nu\, x^\nu\partial_\mu\mathcal{L} = \delta\omega_{\mu\nu}\,\partial^\mu(x^\nu\mathcal{L}). \tag{4.53}$$

Note that the measure $d^{d+1}x$ is Lorentz invariant. Comparing this to an explicit variation of $\mathcal{L}(\phi,\partial\phi)$ implies the relation

$$\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\,\partial^\nu\phi = \frac{\delta\mathcal{L}}{\delta(\partial_\nu\phi)}\,\partial^\mu\phi. \tag{4.54}$$

[12]The anti-symmetry of $\omega_{\mu\nu}$ allows to pull x^ν past the derivative.

This relation holds whenever $\partial_\mu\phi$ appears only in the Lorentz-invariant combination $(\partial\phi)^2 = \eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi)$. For the energy-momentum tensor it implies symmetry in both indices when raised or lowered to the same level

$$T^{\mu\nu} = T^{\nu\mu}. \tag{4.55}$$

The currents $J^{\mu,\rho\sigma} = -J^{\mu,\sigma\rho}$ corresponding to the anti-symmetric matrix $\delta\omega_{\rho\sigma}$ can be expressed in terms of the energy momentum tensor T

$$J^{\mu,\rho\sigma} = T^{\mu\rho}x^\sigma - T^{\mu\sigma}x^\rho. \tag{4.56}$$

Conservation of $J^{\mu,\rho\sigma}$ is then guaranteed by conservation and symmetry of T

$$\partial_\mu J^{\mu,\rho\sigma} = T^{\sigma\rho} - T^{\rho\sigma} = 0. \tag{4.57}$$

The integral of J is the Lorentz angular momentum tensor

$$M^{\mu\nu} = \int d^d\vec{x}\, J^{0,\mu\nu} = \int d^d\vec{x}\,\bigl(T^{0\mu}x^\nu - T^{0\nu}x^\mu\bigr). \tag{4.58}$$

For a scalar field in $d = 3$ dimensional space we obtain the well-known spatial angular momentum

$$J^m = \tfrac{1}{2}\varepsilon^{mkl}M^{kl} = \int d^3\vec{x}\,\dot{\phi}\,((\vec{\partial}\phi)\times\vec{x})^m. \tag{4.59}$$

Furthermore, the momentum for Lorentz boosts reads[13]

$$B^m = M^{0m} = \int d^3\vec{x}\,\bigl(T^{00}x^m\bigr) - P^m t. \tag{4.60}$$

We can also write the Lorentz generators in momentum space[14]

$$M^{\mu\nu} = i\int \frac{d^d\vec{p}}{(2\pi)^d 2e(\vec{p})}\,\bigl(p^\mu\partial^\nu a^*(\vec{p})a(\vec{p}) - p^\nu\partial^\mu a^*(\vec{p})a(\vec{p})\bigr). \tag{4.61}$$

In the quantum theory, all components of the tensor $M^{\mu\nu}$ are hermitian operators. Consequently, the operators for finite transformations are unitary

$$U(\omega) = \exp(\tfrac{i}{2}\omega_{\mu\nu}M^{\mu\nu}), \qquad U(\omega)^\dagger = U(\omega)^{-1}. \tag{4.62}$$

The interesting conclusion is that we have found a unitary representation of the Poincaré group. As the latter is non-compact this representation is necessarily infinite-dimensional. Indeed, the field $\phi(x)$ and Fock space carry infinitely many degrees of freedom.

[13]Conservation basically implies that the motion of the centre of gravity (first term) is governed by the momentum (second term). Note that our convention uses $\vec{p} \sim -m\dot{\vec{x}}$.

[14]Its form is reminiscent of the position space form because Lorentz rotations in both spaces are practically the same

4.3 Poincaré Representations

Above we have derived Lorentz ($M^{\mu\nu}$) and momentum (P^{μ}) generators for relativistic transformations of a scalar field. Let us now discuss the algebraic foundations and generalisations.

Some Basic Definitions. Here are some sketches of basic definitions in group and representation theory.[15]

Group. A set G with an associative composition law $G \times G \to G$ (usually called multiplication), a unit element and inverse map $G \to G$.

Algebra. A vector space A with a bi-linear composition law $A \otimes A \to A$ (usually called multiplication).

Lie Group. A group G that is also a manifold.

Lie Algebra. An algebra g with an anti-symmetric product $[\cdot, \cdot]$ (called Lie bracket) that satisfies the Jacobi identity

$$\big[[a, b], c\big] + \big[[b, c], a\big] + \big[[c, a], b\big] = 0. \tag{4.63}$$

The tangent space of a Lie group G at the unit element is a Lie algebra.

Quantum Group, Quantum Algebra. The algebra of operators in quantum mechanics is called a quantum group or a quantum algebra. In addition to being an algebra, it has a unit element and an inverse for most elements $A^* \to A^*$. It can act as any of the above structures: It is an algebra. It has a subgroup A^*. The subgroup may contain Lie groups. A Lie algebra can be realised by the map $[a, b] = a \cdot b - b \cdot a$ which automatically satisfies the Jacobi identity.

Representation. A map $R : X \to \mathrm{End}(V)$ from a group or an algebra X to linear operators (matrices, endomorphism) on some vector space V. The representation must reflect X's composition law by operator composition (matrix multiplication). If $a \cdot b = c$ then $R(a)R(b) = R(c)$.

Representation of a Lie Algebra. The Lie bracket must be represented by a commutator: If $[a, b] = c$ then $R(a)R(b) - R(b)R(a) = R(c)$.

Physics. The notation in physics often does not distinguish between abstract Lie algebra generators a and their representations $R(a)$, both may be denoted simply by a. Likewise the distinction between Lie brackets and commutators may be dropped (this is perfectly reasonable in a quantum algebra). Moreover the term representation is used not only for an operatorial version of algebra generators, but also for the space on which these operators act (in mathematics: module of the algebra).

Poincaré Algebra. It is straight-forward to derive the algebra of infinitesimal transformations from the operators derived earlier

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\eta^{\nu\rho}M^{\mu\sigma} - i\eta^{\mu\rho}M^{\nu\sigma} - i\eta^{\nu\sigma}M^{\mu\rho} + i\eta^{\mu\sigma}M^{\nu\rho},$$

[15] See a textbook for proper definitions.

$$
\begin{aligned}
[M^{\mu\nu}, P^{\rho}] &= i\eta^{\nu\rho} P^{\mu} - i\eta^{\mu\rho} P^{\nu}, \\
[P^{\mu}, P^{\nu}] &= 0.
\end{aligned}
\tag{4.64}
$$

These define the so-called Poincaré algebra. The operators $M^{\mu\nu}$ generate the algebra so$(d,1)$ of Lorentz (orthogonal) transformations in $d+1$ spacetime dimensions. The spatial components M^{jk} generate the algebra so(d) of rotations in d spatial dimensions.

The Poincaré group is obtained by exponentiating the algebra

$$
g(\omega, a) = \exp\bigl(\tfrac{i}{2}\omega_{\mu\nu} M^{\mu\nu} + i a_{\mu} P^{\mu}\bigr). \tag{4.65}
$$

More precisely it is the component of the Poincaré group connected to the identity element. It includes $\mathrm{Spin}^{+}(d,1)$, the double cover of the proper orthochronous Lorentz group, along with translations.

The above algebra generators and group elements should be viewed as abstract objects in algebra without immediate connection to a physics problem. Above we have found an explicit representation (M,P) of the Poincaré algebra and the corresponding representation $U(\omega,a)$ of the Poincaré group acting on Fock space of a scalar particle. Since (M,P) commute with the number operator N, the representation is reducible.[16]

- The most relevant representation is the one acting on single particle states.
- The other representations are symmetric tensor powers of it.
- The vacuum transform in the trivial representation.
- The single-particle representation is complex, unitary, infinite-dimensional and irreducible.

Unitary Irreducible Representations. We can reproduce what we have learned about the free Klein–Gordon field from representation theory of the Poincaré algebra. We can also learn how to generalise the construction. Let us therefore investigate the unitary irreducible representations of the Poincaré group (Wigner's classification). These will be the elementary building blocks for physical theories with relativistic invariance. The derivation will parallel the derivation of unitary irreducible representations of the rotation group $\mathrm{SO}(3) \simeq \mathrm{SU}(2)$ (in QM1) which leads an understanding of spin. Here the result will characterise the types of admissible particles in a relativistic QFT.

First, we should look for commuting (combinations of) elements of the algebra. Their eigenvalues classify representations because if measured on one state, any other state related to it by symmetry must have the same eigenvalue. The analog for so(3) is the operator J^2. There, a representation of spin j is uniquely characterised by the eigenvalue $j(j+1)$ of J^2.

We notice that the Poincaré algebra possesses a quadratic invariant

$$
P^2 = P^{\mu} P_{\mu}. \tag{4.66}
$$

[16] In other words, the action of (M,P) neither creates nor annihilates particles and will therefore maps $\mathbb{V}_n \to \mathbb{V}_n$. The representation on Fock space thus splits into representations on the individual $\mathbb{V}_n$.

This combination obviously commutes with all the momenta P^μ. It also commutes with the Lorentz generators $M^{\mu\nu}$ because it is constructed as a scalar product.

The combination P^2 must be represented by a unique number on an irreducible representation. Otherwise one could split the representation according to their eigenvalues of P^2. Clearly, P^2 measures the mass of a particle

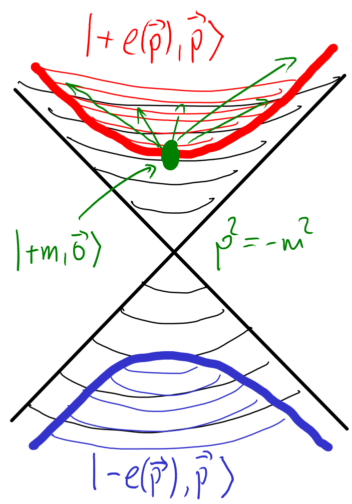
$$P^2 = -m^2. \tag{4.67}$$

For unitary representations P^2 must be real. There are three cases to be distinguished

- $P^2 < 0$, i.e. massive particles,
- $P^2 = 0$, i.e. massless particles,
- $P^2 > 0$, i.e. tachyons.

We shall discuss the massive case in detail and comment only briefly on the others.

The next observation is that the momentum generators P^μ span an abelian ideal[17] of the Poincaré algebra. For abelian ideals the representation space is spanned by simultaneous eigenvectors of all its elements. More concretely, the space is spanned by momentum eigenstates $|p\rangle$ with $P_\mu |p\rangle = p_\mu |p\rangle$. We have already fixed $P^2 = -m^2$ and hence we must restrict to a mass shell $p^2 = -m^2$. As the representation of finite transformations is given by $\exp(ia\cdot P)$, the representation is necessarily complex.



(4.68)

This completes the discussion of the representation of momentum generators P^μ. What about the Lorentz generators $M^{\mu\nu}$?

The condition $p^2 = -m^2$ has two connected components with positive and negative energy, respectively. Orthochronous Lorentz boosts can map between any two momentum vectors on a single mass shell. Mapping between the two mass shells is achieved only by discrete time reversal transformations which we will consider later. For an irreducible representation of the orthochronous Poincaré group, all

[17] An ideal is a subalgebra such that brackets between its elements and elements of the algebra always end up in the subalgebra, here $[M, P] \sim P$.

energies must have the same sign. The positive-energy representation space is now spanned by the vectors $|{+e_m(\vec{p})}, \vec{p}\rangle$. For negative energies we can define states $|{-e_m(\vec{p})}, \vec{p}\rangle$ or alternatively use the hermitian conjugates $|{+e_m(\vec{p})}, \vec{p}\rangle^\dagger \sim \langle {+e_m(\vec{p})}, \vec{p}|$. Let us consider only positive energies from now on; negative energy representations are analogous. All admissible momentum vectors p can be mapped into each other by Lorentz transformations. For further discussions let us restrict to $p = (m, 0)$ which is as good as any other.

Among the Lorentz generators, there are some which change $\vec{p}$. These are the Lorentz boosts. The Lorentz boosts ensure that the following discussion for $p = (m, 0)$ equivalently applies to any other $\vec{p}$.

The transformations which do not change $p = (m, 0)$ are the spatial rotations forming the orthogonal group $\mathrm{SO}(d)$ or its double cover $\mathrm{Spin}(d)$.[18] This group is called the little group (physics) or stabiliser (mathematics) of p. The representation subspace with fixed $\vec{p}$ must therefore transform under a representation of $\mathrm{Spin}(d)$. For the most relevant case of $d = 3$ spatial dimensions, the unitary irreducible representations of $\mathrm{Spin}(3) = \mathrm{SU}(2)$ are labelled by a non-negative half-integer j. Their representation space is spanned by $2j + 1$ vectors $|{-j}\rangle, |{-j-1}\rangle, \ldots, |{+j-1}\rangle, |{+j}\rangle$ with definite z-component of spin. An equivalent representation of $\mathrm{Spin}(d)$ must apply to all momenta $\vec{p}$ because it can be shifted to the point $\vec{p} = \vec{0}$.[19] We have now considered all algebra generators and hence the representation is complete. The representation space is thus spanned by the states $|\vec{p}, j_3\rangle = |\vec{p}\rangle \otimes |j_3\rangle$.

Altogether we find that the massive UIR's of the Poincaré algebra are labelled by their mass $m > 0$, the sign of energy and a unitary irreducible representation of $\mathrm{Spin}(d)$. In the case of $d = 3$, the latter UIR are labelled by a non-negative half-integer j. The representation space for $(m, \pm, j)$ is spanned by the vectors

$$|\vec{p}, j_3\rangle_{(m,\pm,j)} \tag{4.69}$$

with continuous $\vec{p}$ and discrete $j_3 = -j, -j+1, \ldots, j-1, j$.

For spin $j = 0$ the representation space is simply spanned by momentum eigenstates $|\vec{p}\rangle$ with arbitrary three-momentum $\vec{p}$. These are just the single-particle states of a scalar field. The conjugate states $\langle\vec{p}|$ also transform in a UIR, but one with negative momentum.

The next interesting case is $j = \frac{1}{2}$ which we shall discuss in the following section.

In addition, there are massless representations of positive or negative energy. They are classified by a representation of $\mathrm{Spin}(d-1)$.[20] For $d = 3$ the massless representations of $\mathrm{Spin}(2) = \mathrm{U}(1)$ are labelled by a positive or negative half-integer h known as helicity. There is only one state in the representation $(0, \pm, h)$ with

[18] Reflections extend $\mathrm{SO}(d)$ to $\mathrm{O}(d)$ or $\mathrm{Spin}(d)$ to $\mathrm{Pin}(d)$, but they are not included in the identity component of the Poincaré algebra.

[19] Each momentum vector $\vec{p}$ has a different stabiliser subgroup $\mathrm{Spin}(d) \subset \mathrm{Spin}(d, 1)$, but these are all equivalent, and the same applies to their representation.

[20] In fact, the stabiliser is the euclidean group in $d-1$ dimensions which also allows for so-called continuous spin representations.

given helicity

$$|\vec{p}\rangle_{(0,\pm,h)}. \tag{4.70}$$

Last but not least, there is the trivial representation with $P = 0$. The Fock space vacuum transforms under it. Finally, there are tachyonic representations with $P^2 > 0$, but the latter are typically non-unitary.

4.4 Discrete Symmetries

In addition to the continuous symmetries discussed above, there are also relevant discrete symmetries. The most prominent ones are parity, time reversal and charge conjugation. Let us discuss them for the example of a complex scalar field.

Parity. Spatial rotations in d dimensions form the special orthogonal group $\mathrm{SO}(d)$. However, also spatial reflections preserve all distances, and it is natural to consider them among the symmetries, too. Reflections were long believed to be a symmetry of nature, until the electroweak interactions were shown to violate parity symmetry. On the mathematical side, reflections flip the orientation and together with the rotations they form the general orthogonal group $\mathrm{O}(d)$.[21]

For an odd number of spatial dimensions d, it is convenient to introduce parity P as the transformation which inverts all spatial components of the position vector

$$P : (t, \vec{x}) \mapsto (t, -\vec{x}). \tag{4.71}$$

It is an element of $\mathrm{O}(d)$, but not of $\mathrm{SO}(d)$, and it is convenient to choose this element because it does not introduce any preferred directions. There are many more orientation-inverting elements in $\mathrm{O}(d)$; these can be obtained as products of P with elements of $\mathrm{SO}(d)$. Hence it is sufficient to consider only P. In spacetime, introducing parity enlarges the identity component of the Lorentz group $\mathrm{SO}^+(d,1)$ to the orthochronous Lorentz group $\mathrm{O}^+(d,1)$.

A scalar field should transform under parity as follows

$$\begin{aligned} P\phi(t,\vec{x})P^{-1} &= \eta_{\mathrm{P}}\phi(t,-\vec{x}),\\ P\phi^\dagger(t,\vec{x})P^{-1} &= \eta_{\mathrm{P}}^*\phi^\dagger(t,-\vec{x}), \end{aligned} \tag{4.72}$$

where the constant η_{P} is the intrinsic parity of the field ϕ. We want that two parity transformations equal the identity $P^2 = 1$, therefore the parity can be either positive or negative, $\eta_{\mathrm{P}} = \pm 1$.

For the creation and annihilation operators it implies a transformation which reverses the momentum

$$\begin{aligned} Pa(\vec{p})P^{-1} &= \eta_{\mathrm{P}}a(-\vec{p}), & Pa^\dagger(\vec{p})P^{-1} &= \eta_{\mathrm{P}}a^\dagger(-\vec{p}),\\ Pb(\vec{p})P^{-1} &= \eta_{\mathrm{P}}b(-\vec{p}), & Pb^\dagger(\vec{p})P^{-1} &= \eta_{\mathrm{P}}b^\dagger(-\vec{p}). \end{aligned} \tag{4.73}$$

It is a unitary operation.

[21]The double cover of $\mathrm{O}(d)$ is called $\mathrm{Pin}(d)$ in analogy to $\mathrm{Spin}(d)$ which is the double cover of $\mathrm{SO}(d)$.

Time Reversal. The other discrete transformation of the Lorentz group is time reversal

$$T : (t, \vec{x}) \mapsto (-t, \vec{x}). \tag{4.74}$$

It enlarges the orthochronous Lorentz group $\mathrm{O}^+(d, 1)$ to the complete Lorentz group $\mathrm{O}(d, 1)$.

Time reversal is a rather special transformation due to the distinguished role of time in quantum mechanics and special relativity.

For a field ϕ we expect

$$\begin{aligned} T\phi(t, \vec{x})T^{-1} &= \eta_{\mathrm{T}}\phi(-t, \vec{x}), \\ T\phi^\dagger(t, \vec{x})T^{-1} &= \eta_{\mathrm{T}}^*\phi^\dagger(-t, \vec{x}). \end{aligned} \tag{4.75}$$

Comparing to the mode expansion of fields, this could be implemented by a linear transformation of the type $a^\dagger(\vec{p}) \mapsto b(-\vec{p})$. However, such a transformation would not act well on Fock space because it would annihilate all states (but the vacuum).[22] Instead, time reversal (sometimes called motion reversal) is defined by an anti-linear operator which also conjugates plain complex numbers, let us denote it by $\bar{T}$. This inverts the plane wave factors $e^{\pm ip\cdot x}$ and allows to map $a^\dagger \mapsto a^\dagger$, more explicitly

$$\begin{aligned} \bar{T}a(\vec{p})\bar{T}^{-1} &= \eta_{\bar{\mathrm{T}}}^* a(-\vec{p}), & \bar{T}a^\dagger(\vec{p})\bar{T}^{-1} &= \eta_{\bar{\mathrm{T}}} a^\dagger(-\vec{p}), \\ \bar{T}b(\vec{p})\bar{T}^{-1} &= \eta_{\bar{\mathrm{T}}} b(-\vec{p}), & \bar{T}b^\dagger(\vec{p})\bar{T}^{-1} &= \eta_{\bar{\mathrm{T}}}^* b^\dagger(-\vec{p}). \end{aligned} \tag{4.76}$$

The difference w.r.t. parity is merely the anti-linear feature of $\bar{T}$. Time reversal actually allows for a complex $\eta_{\bar{\mathrm{T}}}$ only restricted by $|\eta_{\bar{\mathrm{T}}}|^2 = 1$.

Charge Conjugation. Also the internal symmetry groups can come along with several connected components. For example the complex scalar field has a global $\mathrm{U}(1) = \mathrm{SO}(2)$ symmetry. This can be extended to $\mathrm{O}(2)$ by adding a charge conjugation symmetry.

We already know that the complex conjugate scalar field ϕ^* or $\phi^\dagger$ satisfies the same equations of motion as the original field ϕ. Charge conjugation symmetry thus maps between the fields ϕ and $\phi^\dagger$

$$\begin{aligned} C\phi(x)C^{-1} &= \eta_{\mathrm{C}}\phi^\dagger(x), \\ C\phi^\dagger(x)C^{-1} &= \eta_{\mathrm{C}}^*\phi(x). \end{aligned} \tag{4.77}$$

Requiring that two charge conjugations square to unity, the parity η_{C} must be on the complex unit circle $|\eta_{\mathrm{C}}|^2 = 1$.

$$\begin{aligned} Ca(\vec{p})C^{-1} &= \eta_{\mathrm{C}}^* b(\vec{p}), & Ca^\dagger(\vec{p})C^{-1} &= \eta_{\mathrm{C}} b^\dagger(\vec{p}), \\ Cb(\vec{p})C^{-1} &= \eta_{\mathrm{C}} a(\vec{p}), & Cb^\dagger(\vec{p})C^{-1} &= \eta_{\mathrm{C}}^* a^\dagger(\vec{p}). \end{aligned} \tag{4.78}$$

[22] Unless the vacuum is mapped to a different states, e.g. $\langle 0|$, which makes this definition similar to the conventional one.

Although C maps $\phi \mapsto \phi^\dagger$, it is a perfectly linear map. Charge conjugation is not complex conjugation. One might as well make an anti-linear ansatz for C, but it would lead to a transformation of the kind $a^\dagger \mapsto a$ which would again annihilate almost all of Fock space.

There are several conceptual difficulties with charge conjugation parity:

- In the presence of a corresponding internal symmetry the parity η_{C} actually does not have deeper meaning. In this case one can define a new charge conjugation operation C' by conjugating C by the internal symmetry. This would lead to a different η_{C}, and it makes sense to choose C' such that $\eta_{\mathrm{C}} = 1$.
- Even if there is no continuous internal symmetry, there can be discrete internal symmetries. For example, a possible transformation for a real scalar field is $\phi \mapsto -\phi$.
- In a model with multiple fields, several independent internal parities can coexist, and there may not be a distinguished charge conjugation symmetry. In general, one would expect C to invert all internal charges.
- In the presence of some internal parity C, the spacetime parities P and T become somewhat ambiguous, as one could define $P' = PC$.[23]

Hence the choice of discrete symmetries C, P, T can be ambiguous, and its sometimes tricky to identify the most suitable (or established) one.[24]

Implications. A discrete transformation is a symmetry if it commutes with the Hamiltonian. It is natural to assume parity and time reversal as symmetries of relativistic QFT models and of nature. For a scalar field, it appears impossible to violate parity or time reversal. However, as we shall see, this need not be so for other types of fields. In nature, indeed, some of these symmetries are violated.

Discrete symmetries also lead to conserved charges in the quantum theory. States can be classified by their eigenvalue (parity) under the discrete symmetry. Typically these parities are not additive (as the electrical charge, e.g.), but they only take finitely many values (e.g. $+1$ or -1).

[23] This may appear strange at first sight as P' would conjugate the field ϕ. But by writing $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$ we get two fields with opposite parities η_{P}.

[24] For example, the question of whether a certain discrete symmetry applies should be interpreted as the question where there exist *some choice* of this discrete transformation that is a symmetry.