5 Free Spinor Field

We have seen that next to the scalar field there exist massive representations of Poincaré algebra with spin. The next higher case is spin $j = \frac{1}{2}$. It is described by the Dirac equation, and as a field with half-integer spin it should obey Fermi statistics.

5.1 Dirac Equation and Clifford Algebra

Dirac Equation. Dirac attempted to overcome some of the problems of relativistic quantum mechanics by introducing a first-order wave equation.[1]

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0. \tag{5.1}$$

Here the γ^μ are some suitably chosen operators acting on ψ. This wave equation can be viewed as a factorisation of the second-order Klein–Gordon equation as follows:

$$(i\gamma^\nu \partial_\nu + m)(i\gamma^\mu \partial_\mu - m)\psi = (-\gamma^\nu\gamma^\mu \partial_\nu \partial_\mu - m^2)\psi = 0. \tag{5.2}$$

The latter form becomes the Klein–Gordon equation provided that the γ's satisfy the Clifford algebra[2] [3]

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2\eta^{\mu\nu}. \tag{5.3}$$

This means that every solution of the Dirac equation also satisfies the Klein–Gordon equation and thus describes a particle of mass m.

Clifford Algebra. The Clifford algebra obviously cannot be realised in terms of plain numbers, but finite-dimensional matrices suffice. The realisation of the Clifford algebra strongly depends on the dimension and signature of spacetime.

The simplest non-trivial case is three-dimensional space (without time). A representation of the corresponding Clifford algebra is given by the 2×2 Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.4}$$

[1] The combination of a gamma matrix and an ordinary vector $\gamma^\mu B_\mu$ is often denoted by a slashed vector $\not{B}$.

[2] The indices of the two derivatives are automatically symmetric, hence only the symmetrisation of $\gamma^\mu\gamma^\nu$ must equal $-\eta^{\mu\nu}$.

[3] We will use conventional γ matrices for signature $+---$ and the minus sign in the Clifford algebra adjusts for our choice of opposite signature. Alternatively, one could multiply all γ-matrices by i and drop the minus sign.

One can convince oneself that these matrices obey the algebra[4]

$$\sigma^j \sigma^k = \delta^{jk} + i\varepsilon^{jkl}\sigma^l. \tag{5.5}$$

This also implies the three-dimensional Clifford algebra

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij}. \tag{5.6}$$

In this course we will be predominantly be interested in the case of $d = 3$ spatial dimensions plus time, i.e. spacetime with $D = d + 1 = 4$ dimensions. There, the smallest non-trivial representation of the Clifford algebra is four-dimensional (coincidence!). The elements of this four-dimensional vector space are called spinors, more precisely, Dirac spinors or 4-spinors.

There are many equivalent ways to write this representation as 4×4 matrices. The best-known ones are the Dirac, Weyl and Majorana representations. These are often presented in the form of 2×2 matrices whose elements are again 2×2 matrices. The latter are written using the Dirac matrices. We shall mainly use the Weyl representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^k = \begin{pmatrix} 0 & +\sigma^k \\ -\sigma^k & 0 \end{pmatrix}. \tag{5.7}$$

One can easily confirm that these matrices obey the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$ by means of the three-dimensional Clifford algebra. A useful property of the Weyl representation is that all four gamma matrices are block off-diagonal. The Dirac and Majorana representations have different useful properties. In most situations, it is however convenient not to use any of the explicit representations, but work directly with the abstract Clifford algebra.

Solutions. The Dirac equation is homogeneous therefore it is conveniently solved by Fourier transformation

$$\psi(x) = \int d^4p\, e^{-ip\cdot x}\psi(p), \qquad (p_\mu\gamma^\mu - m)\psi = 0. \tag{5.8}$$

To construct the solutions, let us introduce the matrices

$$\mathit{\Pi}^\pm = \frac{1}{2m}(m \pm p\cdot\gamma) \tag{5.9}$$

such that the Dirac equation becomes $\mathit{\Pi}^-\psi = 0$. We are interested in the kernel of $\mathit{\Pi}^-$.

As noted above, we have the identity

$$\mathit{\Pi}^+\mathit{\Pi}^-\psi = \frac{1}{4m^2}\,(m^2 + p^2)\psi. \tag{5.10}$$

[4]We will not introduce a distinguished symbol for unit matrices. 1 is the unit element. Here the term δ^{ij} has an implicit 2×2 unit matrix.

This operators acts identically on all components of ψ. Any solution therefore requires the mass shell condition $p^2 = -m^2$.

On the mass shell $p^2 = -m^2$, the operators $\Pi^\pm$ act as orthogonal projectors:

$$\Pi^\pm \Pi^\pm = \Pi^\pm, \qquad \Pi^\pm \Pi^\mp = 0. \tag{5.11}$$

Now the operators $\Pi^\pm$ are very similar.[5] Evidently, their kernels have the same dimension. Therefore $\Pi^\pm$ both have half-maximal rank. The Dirac equation therefore has two solutions for each on-shell momentum p.

A basis of two positive-energy solutions is denoted by

$$u_a(\vec{p}), \quad a = \pm, \qquad (p{\cdot}\gamma - m)u_a(\vec{p}) = 0. \tag{5.12}$$

Instead of introducing negative-energy solutions, we prefer to consider equivalent positive-energy solutions of the opposite Dirac equation $\Pi^+\psi = 0$

$$v_a(\vec{p}), \quad a = \pm, \qquad (p{\cdot}\gamma + m)v_a(\vec{p}) = 0. \tag{5.13}$$

To write such solutions explicitly, we can recycle the projectors $\Pi^\pm$ and set

$$u = \Pi^+\lambda, \qquad v = \Pi^-\lambda, \tag{5.14}$$

where λ is some spinor. The properties of the projectors immediately show that u and v are solutions to their respective equations. Note, however, that some components of λ are projected out in u and in v.

Let us consider explicitly solutions in the Weyl representation. E.g. setting $\lambda = (\kappa, 0)$ with κ some 2-spinor, we find

$$\begin{aligned} u(\vec{p}) &= \frac{1}{2m}\begin{pmatrix} m & e(\vec{p}) + \vec{p}{\cdot}\vec{\sigma} \\ e(\vec{p}) - \vec{p}{\cdot}\vec{\sigma} & m \end{pmatrix}\begin{pmatrix} \kappa \\ 0 \end{pmatrix} \\ &= \frac{1}{2m}\begin{pmatrix} m\kappa \\ e(\vec{p})\kappa - \vec{p}{\cdot}\vec{\sigma}\kappa \end{pmatrix}, \\ v(\vec{p}) &= \frac{1}{2m}\begin{pmatrix} m\kappa \\ -e(\vec{p})\kappa + \vec{p}{\cdot}\vec{\sigma}\kappa \end{pmatrix}. \end{aligned} \tag{5.15}$$

There are two independent choices for the 2-spinor κ, hence there are two solutions for u and v, respectively. One typically considers $u_\gamma(\vec{p})$, $v_\gamma(\vec{p})$, $\gamma = \pm$, as two pairs of fixed basis vectors for each momentum $\vec{p}$.

Altogether the general solution can now be expanded as

$$\psi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3\, 2e(\vec{p})}\, \bigl(e^{-ip\cdot x}u_\gamma(\vec{p})b_\gamma(\vec{p}) + e^{ip\cdot x}v_\gamma(\vec{p})a^\dagger_\gamma(\vec{p})\bigr) \tag{5.16}$$

Here the negative-energy coefficient b_γ is chosen differently from a_γ because gamma matrices are generally complex and therefore also the Dirac spinor ψ.

[5] We might as well have declared $(i\gamma^\mu\partial_\mu + m)\psi = 0$ to be the Dirac equation.

5.2 Poincaré Symmetry

The Dirac equation is a relativistic wave equation. Translational invariant is evident. Most importantly, we have not yet shown its Lorentz covariance (although the resulting Klein–Gordon equation certainly is covariant).

Lorentz Symmetry. Let us therefore consider a Lorentz transformation $x' = \Lambda^{-1}x$ with $\Lambda(\omega) = \exp(\omega)$. Suppose ψ is a solution of the Dirac equation. It is not sufficient to use the transformation rule for scalar fields $\psi'(x') = \psi(x)$. In analogy to vectors we should also transform spinors. We make the ansatz

$$\psi'(x') = S(\omega)\psi(x), \tag{5.17}$$

where $S(\omega)$ is a matrix that acts on Dirac spinors. We then substitute $\psi'(x) = S\psi(\Lambda x)$ into the Dirac equation

$$\begin{aligned}
0 &= i\gamma^\mu \partial_\mu \psi' - m\psi' \\
&= \big(i\gamma^\nu S \Lambda^\mu{}_\nu \partial_\mu \psi - Sm\psi\big)(\Lambda x) \\
&= S\big(iS^{-1}\gamma^\nu S \Lambda^\mu{}_\nu \partial_\mu \psi - i\gamma^\mu \partial_\mu \psi\big)(\Lambda x) \\
&= iS\big(\Lambda^\mu{}_\nu S^{-1}\gamma^\nu S - \gamma^\mu\big)(\partial_\mu \psi)(\Lambda x).
\end{aligned} \tag{5.18}$$

So the term in the bracket must vanish for invariance of the Dirac equation.

Indeed, the canonical Lorentz transformation of gamma matrices

$$\gamma'^\mu = (\Lambda^{-1})^\mu{}_\nu \, S\gamma^\nu S^{-1}, \tag{5.19}$$

where not only the vector index is transformed by Λ^{-1}, but also the spinor matrix is conjugated by the corresponding spinor transformation S.[6] In analogy to the invariance of the Minkowski metric, $\eta' = \eta$, the Dirac equation is invariant if the gamma matrices are invariant

$$\gamma'^\mu = \gamma^\mu. \tag{5.20}$$

This condition relates S to the Lorentz transformation Λ.

The infinitesimal form of the invariance condition reads

$$[\delta S, \gamma^\mu] - \delta\omega^\mu{}_\nu \gamma^\nu = 0, \tag{5.21}$$

Now δS does not carry any vector indices, but it should be proportional to two $\delta\omega_{\mu\nu}$ which carries two of them. We can only contract them to two gamma matrices, and we make the ansatz $\delta S = \frac{1}{2}\alpha\delta\omega_{\mu\nu}\gamma^\mu\gamma^\nu$. Substituting this into the invariance condition and using

$$[\gamma^\rho\gamma^\sigma, \gamma^\mu] = \gamma^\rho\{\gamma^\sigma, \gamma^\mu\} - \{\gamma^\rho, \gamma^\mu\}\gamma^\sigma. \tag{5.22}$$

we arrive at $(2\alpha - 1)\delta\omega^\mu{}_\nu\gamma^\nu = 0$. We conclude that a Lorentz transformation for spinors is given by the matrix

$$\delta S = \tfrac{1}{4}\delta\omega_{\mu\nu}\gamma^\mu\gamma^\nu \quad \text{or} \quad S(\omega) = \exp\big(\tfrac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu\big). \tag{5.23}$$

[6] In general, spinors are transformed by S from the left, co-spinors by the S^{-1} from the right.

Comparing this result to the abstract form of finite Lorentz transformations as $U(\omega) = \exp(\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu})$ we have derived a new representation on spinors

$$M^{\mu\nu} = -\tfrac{i}{4}[\gamma^\mu, \gamma^\nu]. \tag{5.24}$$

This representation obeys the Lorentz algebra derived above, i.e. $[M^{\mu\nu}, M^{\rho\sigma}] = iM + \ldots$.

Double Cover. Spinor representations exist only for the double cover $\mathrm{Spin}(X)$ of an orthogonal group $\mathrm{SO}(X)$. Let us observe this fact in a simple example.

Consider a rotation in the x-y-plane with angle $\omega_{12} = -\omega_{21} = \varphi$. The associated finite Lorentz transformation matrix in the x-y-plane reads

$$\Lambda(\varphi) = \exp(\omega) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}. \tag{5.25}$$

The associated spinor transformation reads

$$S(\varphi) = \operatorname{diag}(e^{-i\varphi/2}, e^{+i\varphi/2}, e^{-i\varphi/2}, e^{+i\varphi/2}). \tag{5.26}$$

The vector rotation $\Lambda(\varphi)$ is 2π-periodic in φ whereas the spinor rotation is merely 4π-periodic. The rotation by $\varphi = 2\pi$ is represented by the unit matrix for vectors, but for spinors it is the negative unit matrix

$$\Lambda(2\pi) = 1 = (-1)^F, \qquad S(2\pi) = -1 = (-1)^F. \tag{5.27}$$

The spin group thus has an element which represents a rotation by 2π (irrespectively of the direction). On vector representations (integer spin) it acts as the identity, on spinor representations (half-integer spin) it acts as -1. Due to the relation between spin and statistics, the extra element is equivalent to $(-1)^F$ where F measures the number of fermions (odd for spinors, even for vectors).

Chiral Representation. There is an important feature of the spin representation $M^{\mu\nu}$ which is best observed in the Weyl representation of gamma matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}. \tag{5.28}$$

Here we have introduced the sigma matrices σ^μ, $\bar\sigma^\mu$ as an extension of the Pauli matrices σ^k to four spacetime dimensions as follows

$$\sigma^0 = \bar\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \bar\sigma^k = -\sigma^k. \tag{5.29}$$

The Lorentz representation now reads

$$M^{\mu\nu} = -\frac{i}{4}\begin{pmatrix} \sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu & 0 \\ 0 & \bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu \end{pmatrix}. \tag{5.30}$$

This representation has block-diagonal form and therefore reduces to two independent representations $M^{\mu\nu} = \operatorname{diag}(M_{\mathrm{L}}^{\mu\nu}, M_{\mathrm{R}}^{\mu\nu})$ with

$$M_{\mathrm{L}}^{\mu\nu} = -\tfrac{i}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu), \qquad M_{\mathrm{R}}^{\mu\nu} = -\tfrac{i}{4}(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu). \tag{5.31}$$

In other words, the Dirac spinor $\psi = (\psi_{\mathrm{L}}, \psi_{\mathrm{R}})$ transforms in the direct sum of two (irreducible) representations of the Lorentz group. The 2-spinors ψ_{L} and ψ_{R} are called left-chiral and right-chiral spinors. The massive Dirac equation, however, mixes these two representations

$$\begin{aligned} i\sigma^\mu\partial_\mu\psi_{\mathrm{R}} - m\psi_{\mathrm{L}} &= 0, \\ i\bar\sigma^\mu\partial_\mu\psi_{\mathrm{L}} - m\psi_{\mathrm{R}} &= 0. \end{aligned} \tag{5.32}$$

It is therefore convenient to use Dirac spinors for massive spinor particles. We shall discuss the massless case later on.

The decomposition into chiral parts is not just valid in the Weyl representation of the Clifford algebra. More abstractly, it is due to the existence of the matrix

$$\gamma^5 = \tfrac{i}{24}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{5.33}$$

In the Weyl representation it reads $\gamma^5 = \operatorname{diag}(-1, +1)$, it therefore measures the chirality of spinors. In general, it anti-commutes with all the other gamma matrices,

$$\{\gamma^5, \gamma^\mu\} = 0. \tag{5.34}$$

This property implies that a single gamma matrix maps between opposite chiralities, i.e. it inverts chirality. The property is also sufficient to prove commutation with $M^{\mu\nu}$. Alternatively, it follows by construction of γ^5 as a (pseudo)-scalar combination of gamma matrices.

A further useful property is

$$\gamma^5\gamma^5 = 1, \tag{5.35}$$

It can be used to show that the combinations $\frac{1}{2}(1 \pm \gamma_5)$ are two orthogonal projectors to the chiral subspaces.

Sigma Matrices. Let us briefly discuss the sigma matrices which are chiral analogs of the gamma matrices. The sigma matrices obey an algebra reminiscent of the Clifford algebra[7] [8]

$$\sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu = -2\eta^{\mu\nu} = \bar\sigma^\mu\sigma^\nu + \bar\sigma^\nu\sigma^\mu. \tag{5.36}$$

Inspection shows that all sigma matrices are hermitian

$$(\sigma^\mu)^\dagger = \sigma^\mu, \qquad (\bar\sigma^\mu)^\dagger = \bar\sigma^\mu. \tag{5.37}$$

[7] The assignment of bars enables a 2-dimensional representation for this algebra unlike the Clifford algebra which requires a larger 4-dimensional representation.

[8] Note that for any reasonable product of sigma matrices the sequence of factors will alternate between σ and $\bar\sigma$. This agrees with the fact that a single γ maps between the two chiralities.

In other words, the four sigma matrices form a real basis for 2×2 hermitian matrices. Likewise one can confirm that that the matrices $M^{\mu\nu}_{\mathrm{L,R}}$ form a real basis of 2×2 complex traceless matrices.[9] Furthermore,

$$(M^{\mu\nu}_{\mathrm{L}})^\dagger = -M^{\mu\nu}_{\mathrm{R}}. \tag{5.38}$$

These are just the defining relations for the fundamental representation of $\mathrm{sl}(2,\mathbb{C})$ along with its conjugate representation.[10]

The Lorentz algebra $\mathrm{so}(3,1)$ is indeed equivalent to the algebra $\mathrm{sl}(2,\mathbb{C})$. At the level of groups, $\mathrm{Spin}^+(3,1) = \mathrm{SL}(2,\mathbb{C})$ is the double cover of $\mathrm{SO}^+(3,1)$.

- A chiral 2-spinor of $\mathrm{Spin}^+(3,1)$ is equivalent to the fundamental representation (space) of $\mathrm{SL}(2,\mathbb{C})$.
- Similarly, a 2-spinor of opposite chirality is equivalent to the conjugate fundamental representation (space) of $\mathrm{SL}(2,\mathbb{C})$.
- Spinor representation exist only for the double-cover group $\mathrm{Spin}^+(3,1)$, but not for the original Lorentz group $\mathrm{SO}^+(3,1)$.

5.3 Discrete Symmetries

In addition to the continuous Poincaré symmetry and an obvious U(1) internal symmetry, there are several discrete symmetries and transformations which we shall now discuss. These are also needed to formulate a Lagrangian.

Parity. Spatial parity $\vec{x}\,' = -\vec{x}$ is the simplest discrete symmetry. We make the usual ansatz

$$\psi'(t,-\vec{x}) = \gamma_{\mathrm{P}}\psi(t,\vec{x}), \tag{5.39}$$

where γ_{P} is a matrix that induces the reflection on spinors.

The new field obeys the same old Dirac equation provided that the gamma matrices are invariant

$$\gamma'^\mu := \Lambda^\mu{}_\nu \gamma_{\mathrm{P}} \gamma^\nu \gamma_{\mathrm{P}}^{-1} \stackrel{!}{=} \gamma^\mu. \tag{5.40}$$

We need to find a matrix γ_{P} that

- commutes with γ^0 (because $\Lambda^0{}_0 = 1$),
- anti-commutes with γ^k (to compensate $\Lambda^k{}_k = -1$),
- squares to unity (because $P^2 = 1$).

This matrix is easily identified as

$$\gamma_{\mathrm{P}} = \gamma^0. \tag{5.41}$$

Note that γ_{P} interchanges the two chiralities. Hence the Dirac spinor is

[9] A traceless 2×2 matrix has $2 \cdot 2 - 1 = 3$ degrees of freedom. If the latter are complex, there are altogether 6 real d.o.f.. A pair of anti-symmetric vector indices provides the same number of d.o.f., noting that all $M^{\mu\nu}_{\mathrm{L,R}}$ are linearly independent (over the real numbers).

[10] The group $\mathrm{SL}(N)$ of matrices with unit determinant is generated by the algebra $\mathrm{sl}(N)$ of traceless matrices.

- reducible under proper orthochronous Lorentz rotations,
- but irreducible under orthochronous Lorentz rotations.

Time Reversal. Anti-linear time (motion) reversal also has a representation on spinors

$$\psi'(-t,\vec{x}) = \gamma_{\bar{\mathrm{T}}}\psi(t,\vec{x}), \tag{5.42}$$

The anti-linear nature of $\bar{T}$ implies that a solutions of the Dirac equation should be mapped to a solution of the complex conjugated Dirac equation. In the Weyl representation this is achieved by the matrix

$$\gamma_{\bar{\mathrm{T}}} = \gamma^1\gamma^3. \tag{5.43}$$

The gamma matrices satisfy the following identity with the time reversal matrix

$$\Lambda^\mu{}_\nu \gamma_{\bar{\mathrm{T}}}(\gamma^\nu)^*\gamma_{\bar{\mathrm{T}}}^{-1} = -\gamma^\mu. \tag{5.44}$$

Charge Conjugation. The Dirac field is charged, it therefore makes sense to define charge conjugation. We will use it later to investigate the statistics associated to spinor fields.

Linear charge conjugation maps a field to its conjugate field[11]

$$\psi'(x) = \gamma_{\mathrm{C}}\psi^{\dagger\mathrm{T}} \tag{5.45}$$

such that ψ' solves the same wave equation as ψ. Let us substitute

$$(i\gamma^\mu\partial_\mu - m)\psi' = (i\gamma^\mu\partial_\mu - m)\gamma_{\mathrm{C}}\psi^{\dagger\mathrm{T}} = \big((-i(\gamma^\mu)^*\partial_\mu - m)\gamma_{\mathrm{C}}^*\psi\big)^{\dagger\mathrm{T}}. \tag{5.46}$$

This vanishes if

$$\gamma_{\mathrm{C}}(\gamma^\mu)^*\gamma_{\mathrm{C}}^{-1} = -\gamma^\mu. \tag{5.47}$$

In the Weyl representation only γ^2 is imaginary, and the condition is solved by the matrix

$$\gamma_{\mathrm{C}} = -i\gamma^2. \tag{5.48}$$

CPT-Transformation In QFT a discrete transformation of fundamental importance is the combination of charge conjugation, parity and time reversal, called CPT. Effectively, it flips the sign of all coordinates[12] and performs a complex conjugation.

A spinor transforms according to

$$\psi'(x) = \gamma_{\bar{\mathrm{T}}}\gamma_{\mathrm{P}}\gamma_{\mathrm{C}}\psi^{\dagger\mathrm{T}}(-x) \tag{5.49}$$

[11] The composition of adjoint and transpose operations is almost the same as complex conjugation. There is however a slight difference which becomes relevant only later.

[12] This is an orientation-preserving transformation which belongs to $\mathrm{Spin}(3,1)$, but not to $\mathrm{Spin}^+(3,1)$.

We find that the combination of matrices is just the additional gamma matrix γ^5

$$\gamma_{\bar{\mathrm{T}}}\gamma_{\mathrm{P}}\gamma_{\mathrm{C}} = -i\gamma^1\gamma^3\gamma^0\gamma^2 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5. \tag{5.50}$$

This anti-commutes with all gamma matrices

$$\gamma^5\gamma^\mu\gamma^5 = -\gamma^\mu. \tag{5.51}$$

The sign is compensated by flipping the sign of all vectors.

The CPT-theorem states that all reasonable relativistic QFT's must be invariant under the CPT-transformation. They need not be invariant under any of the individual transformations.

Hermitian Conjugation. The Dirac spinor ψ is complex. To construct real quantities for use in the Lagrangian or the Hamiltonian one typically uses hermitian conjugation. However, the various gamma matrices transform differently under this operation.

The transformation can be uniformised by conjugation with some other matrix $\gamma_\dagger$

$$\gamma_\dagger(\gamma^\mu)^\dagger\gamma_\dagger^{-1} = \gamma^\mu. \tag{5.52}$$

In most relevant representations, in particular in the chiral one, one finds

$$\gamma_\dagger = \gamma_\dagger^{-1} = \gamma^0. \tag{5.53}$$

Therefore, one should modify hermitian conjugation for a spinor ψ and likewise for a spinor matrix X as

$$\bar\psi = \psi^\dagger\gamma_\dagger^{-1}, \qquad \bar X = \gamma_\dagger X\gamma_\dagger^{-1}. \tag{5.54}$$

The gamma matrices are self-adjoint under modified hermitian conjugation, $\bar\gamma^\mu = \gamma^\mu$.

5.4 Spin Statistics

So far we have only considered the Dirac equation. For quantisation, conserved charges and later for adding interactions we should construct a Lagrangian.

Lagrangian. It is straight-forward to guess

$$\mathcal{L} = \bar\psi(i\gamma^\mu\partial_\mu - m)\psi. \tag{5.55}$$

The variation w.r.t. $\psi^\dagger$ obviously yields the Dirac equation. Variation w.r.t. ψ gives the hermitian conjugate equation

$$-i\partial_\mu\bar\psi\gamma^\mu - m\bar\psi = (i\gamma^\mu\partial_\mu\psi - m\psi)^\dagger\gamma^0 = 0 \tag{5.56}$$

In fact, the Lagrangian is almost real

$$\begin{aligned}\mathcal{L}^\dagger &= \psi^\dagger(-i(\gamma^\mu)^\dagger\partial_\mu^\dagger - m)\gamma^0\psi = \bar\psi(-i\gamma^\mu\partial_\mu^\dagger - m)\psi \\ &= -i\partial_\mu(\bar\psi\gamma^\mu\psi) + \mathcal{L}.\end{aligned} \tag{5.57}$$

There is merely an imaginary topological term left,[13] the action is manifestly real.

Hamiltonian Formulation. To go to the Hamiltonian framework we compute the conjugate momenta $\pi = \partial\mathcal{L}/\partial\dot\psi = i\psi^\dagger$ and $\pi^\dagger = \partial\mathcal{L}/\partial\dot\psi^\dagger = 0$. It turns out that the conjugate momenta are proportional to the fields:[14]

- Dirac equation is a first-order differential equation.
- There are no independent momenta.
- Phase space equals position space.

Keep in mind that ψ and $\psi^\dagger$ are canonically conjugate fields.

Instead of computing the Hamiltonian, we can compute the energy-momentum tensor of which it is a component

$$T^{\mu\nu} = i\bar\psi\gamma^\mu\partial^\nu\psi - \eta^{\mu\nu}\mathcal{L}. \tag{5.58}$$

Unfortunately this tensor is not symmetric as it should for Lorentz invariance. Gladly, the anti-symmetric part can be written as (making use of the e.o.m.)

$$T^{[\mu\nu]} = i\partial_\rho(\bar\psi\gamma^{[\rho}\gamma^\mu\gamma^{\nu]}\psi) = \partial_\rho K^{\rho\mu\nu}. \tag{5.59}$$

The contribution from K is a boundary term for the integral defining the total momentum integral P^μ. We should thus subtract $\partial_\rho K^{\rho\mu\nu}$ from $T^{\mu\nu}$.

The Hamiltonian for the Dirac equation now reads

$$H = -\int d^3x\, T^{00} = \int d^3x\, \bar\psi\big(-i\vec\gamma\cdot\vec\partial + m\big)\psi. \tag{5.60}$$

Charge Conjugation. A conventional treatment and conventional quantisation of the above framework of the Dirac equation leads to several undesirable features.

For example, the problem manifests for charge conjugation. For every solution ψ of the Dirac equation, there is a charge conjugate solution $\psi_\mathrm{C} = \gamma_\mathrm{C}\psi^{\dagger\mathrm{T}}$. Let us

[13] The topological term can be removed from the Lagrangian to obtain a manifestly real $\mathcal{L}' = \mathcal{L} - \frac{i}{2}\partial_\mu(\bar\psi\gamma^\mu\psi)$.

[14] Here, $\pi^\dagger$ is not the complex conjugate of π because $\mathcal{L}$ is not real. For the real $\mathcal{L}'$ we get instead $\pi = \frac{i}{2}\psi^\dagger$ and $\pi^\dagger = -\frac{i}{2}\psi$.

compute its energy

$$\begin{aligned}
H[\psi_{\mathrm{C}}] &= \int d^3x\, \psi^{\mathrm{T}}(\gamma_{\mathrm{C}})^\dagger\gamma^0\big(-i\vec{\gamma}\cdot\vec{\partial}+m\big)\gamma_{\mathrm{C}}\psi^{\dagger\mathrm{T}}\\
&= \int d^3x\, \psi^{\mathrm{T}}(\gamma_{\mathrm{C}})^\dagger\gamma^0\gamma_{\mathrm{C}}\big(+i(\vec{\gamma})^*\cdot\vec{\partial}+m\big)\psi^{\dagger\mathrm{T}}\\
&= -\int d^3x\, \psi^{\mathrm{T}}\gamma^0\big(+i(\vec{\gamma})^*\cdot\vec{\partial}+m\big)\psi^{\dagger\mathrm{T}}\\
&= -\int d^3x\, \psi^{\mathrm{T}}\big(+i(\vec{\gamma})^{\mathrm{T}}\cdot\vec{\partial}+m\big)\gamma^0\psi^{\dagger\mathrm{T}}\\
&\overset{*}{=} -\int d^3x\, \psi^\dagger\gamma^0\big(+i\vec{\gamma}\cdot\vec{\partial}^{\,\mathrm{T}}+m\big)\psi\\
&= -\int d^3x\, \bar{\psi}\big(-i\vec{\gamma}\cdot\vec{\partial}+m\big)\psi = -H[\psi]
\end{aligned} \tag{5.61}$$

The charge conjugate solution has opposite energy

$$H[\psi_{\mathrm{C}}] = -H[\psi]. \tag{5.62}$$

Compare to the scalar field and $\phi_{\mathrm{C}} = \phi^*$. There we have

$$H[\phi_{\mathrm{C}}] = +H[\phi]. \tag{5.63}$$

This is consistent with positivity of the energy. Naively, the field ψ could not possibly have positive definite energy.

Related issues arise for propagators and causality.

All above steps are elementary. Only one step $(*)$ can be changed: Transposition. We have used

$$\psi^{\mathrm{T}}X\psi^{\dagger\mathrm{T}} = \psi^a X_a{}^b\psi^\dagger_b = \psi^\dagger_b X_a{}^b\psi^a = \psi^\dagger X^{\mathrm{T}}\psi. \tag{5.64}$$

Instead of $\psi^a\psi^\dagger_b = \psi^\dagger_b\psi^a$ could use a different rule [15]

$$\psi^a\psi^\dagger_b = -\psi^\dagger_b\psi^a. \tag{5.65}$$

This inserts a minus sign at $*$ and the energy of a solution and its charge conjugate are the same

$$H[\psi_{\mathrm{C}}] = +H[\psi]. \tag{5.66}$$

This solves all the issues of the spinor field.

Spin-Statistics Theorem. The spin-statistics theorem states that consistent quantisation of fields with half-integer spin requires the use of anti-commutation relations

$$\{\psi, \psi^\dagger\} \sim \hbar. \tag{5.67}$$

Such fields are called fermionic, they obey the Fermi-Dirac statistics.
Multi-particle wave functions will be totally anti-symmetric.

[15] Eventually, quantisation will make ψ's become operators which do not commute either.

Conversely, fields with integer spin require commutation relations

$$[\phi, \phi^\dagger] \sim \hbar. \tag{5.68}$$

These fields are called bosonic, they obey the Bose-Einstein statistics. Multi-particle wave functions will be totally symmetric.

5.5 Grassmann Numbers

Quantisation should be viewed as a deformation of classical physics. Therefore, the anti-commutation relations of the quantum theory $\{\psi, \psi^\dagger\} \sim \hbar$ should be reflected by anti-commuting fields $\{\psi, \psi^\dagger\} = 0$ in the classical theory. More generally,

$$\psi^a \psi^b = -\psi^b \psi^a, \qquad \psi^a \psi^\dagger_b = -\psi^\dagger_b \psi^a, \qquad \psi^\dagger_a \psi^\dagger_b = -\psi^\dagger_b \psi^\dagger_a. \tag{5.69}$$

Besides these additional signs, the fields ψ will commute with numbers and scalar fields.

We therefore cannot use ordinary commuting numbers to represent the field ψ in the classical Lagrangian, we need something else.

Description. The required extension of the concept of numbers is called Grassmann numbers:

- Grassmann numbers form a non-commutative ring with $\mathbb{Z}_2$ grading.
- Grassmann numbers are $\mathbb{Z}_2$-graded, they can be even or odd: $|a| = 0, 1$.[16]
- Sums and products respect the even/odd grading

$$|a + b| = |a| = |b|, \qquad |ab| = |a| + |b|. \tag{5.70}$$

- The product is commutative unless both factors are odd in which case it is anti-commutative

$$ab = (-1)^{|a||b|} ba. \tag{5.71}$$

- Ordinary numbers are among the even Grassmann numbers.
- The field ψ takes values in odd Grassmann numbers.
- Real and complex Grassmann numbers can be defined. Grassmann numbers then form an algebra over the respective field.

A basis a^n of odd Grassmann numbers can be constructed out of a Clifford algebra $\{\gamma^j, \gamma^k\} = 2\delta^{jk}$

$$a^n = \frac{1}{\sqrt{2}} \big(\gamma^{2n} + i\gamma^{2n+1}\big). \tag{5.72}$$

In other words, Grassmann numbers can be represented in terms of (large) matrices. One should view the basis a^n to be sufficiently large or infinite.[17]

[16] Linear combinations of even and odd numbers could be defined, but usually they do not appear.

[17] There is no distinguished element such as i which extends the real numbers to complex numbers. We therefore do not have universal means to assign a value to a Grassmann variable. We will mainly use Grassmann variables to describe classical (fermionic) fields without assigning values.

Calculus. One can do calculus with Grassmann numbers much like ordinary numbers, but note:

- odd numbers square to zero: $(a^n)^2 = \frac{1}{2}\{a^n, a^n\} = 0$.
- the square root of zero is ill-defined.
- odd numbers have no inverse.
- some even numbers (e.g. products of two odd numbers) have no inverse.

A derivative for odd numbers can be defined as usual

$$\frac{\partial}{\partial a^m} a^n = \delta^n_m. \tag{5.73}$$

Note that derivatives are also odd objects

$$\{\partial/\partial a^m, \partial/\partial a^n\} = 0. \tag{5.74}$$

The above relation should be written as

$$\{\partial/\partial a^m, a^n\} = \delta^n_m. \tag{5.75}$$

We can also define the derivatives as elements of the same Clifford algebra

$$\frac{\partial}{\partial a^n} = \frac{1}{\sqrt{2}}\bigl(\gamma^{2n} - i\gamma^{2n-1}\bigr). \tag{5.76}$$

This leads to the same anti-commutation relations as above.

Complex Conjugation. A complex Grassmann number a can be written as a combination of the real Grassmann numbers $a_\mathrm{r}, a_\mathrm{i}$ as

$$a = a_\mathrm{r} + i a_\mathrm{i}. \tag{5.77}$$

Spinor fields are typically complex and we often need to complex conjugate them. Confusingly, there are two equivalent definitions of complex conjugation for Grassmann numbers.

One is reminiscent of complex conjugation

$$a^* = a_\mathrm{r} - i a_\mathrm{i}. \tag{5.78}$$

It obviously satisfies

$$(ab)^* = a^* b^*. \tag{5.79}$$

The other conjugation is reminiscent of hermitian adjoint. It satisfies

$$(ab)^\dagger = b^\dagger a^\dagger. \tag{5.80}$$

For ordinary numbers it would be the same as complex conjugation, but odd Grassmann numbers do not commute. The two definitions can be related as follows

$$a^\dagger = \begin{cases} a^* & \text{if } a \text{ is even,} \\ -ia^* & \text{if } a \text{ is odd.} \end{cases} \tag{5.81}$$

QM and QFT frequently use the adjoint operation, hence it is convenient to also use the adjoint for Grassmann numbers.

One should pay attention in defining real and odd Grassmann numbers. An odd number which satisfies $a^\dagger = a$ is not real. In particular, the even product of two such numbers is imaginary

$$(ab)^\dagger = b^\dagger a^\dagger = ba = -ab. \tag{5.82}$$

Instead real odd numbers are defined by

$$a^\dagger = -ia^* = -ia. \tag{5.83}$$

5.6 Quantisation

Poisson Brackets. We have also seen that ψ and $\psi^\dagger$ are canonically conjugate fields, there is no need to introduce additional conjugate momenta. The Poisson bracket for the spinor field should read[18]

$$\{F, G\} = i \int d^3x \left(\frac{\delta F}{\delta\psi^a(\vec{x})} \frac{\delta G}{\delta\psi^\dagger_a(\vec{x})} + \frac{\delta F}{\delta\psi^\dagger_a(\vec{x})} \frac{\delta G}{\delta\psi^a(\vec{x})} \right). \tag{5.84}$$

This expression can also be written as[19]

$$\{\psi^a(\vec{x}), \psi^\dagger_b(\vec{y})\} = \{\psi^\dagger_b(\vec{y}), \psi^a(\vec{x})\} = i\delta^a_b \delta^3(\vec{x} - \vec{y}). \tag{5.85}$$

Anti-Commutators. For quantisation, these Poisson brackets are replaced by an anti-commutator[20]

$$\{\psi^a(\vec{x}), \psi^\dagger_b(\vec{y})\} = \delta^a_b \delta^3(\vec{x} - \vec{y}). \tag{5.86}$$

By Fourier transformation to momentum space

$$\psi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3\, 2e(\vec{p})} \left(e^{-ip\cdot x} u_\alpha(\vec{p}) b_\alpha(\vec{p}) + e^{ip\cdot x} v_\alpha(\vec{p}) a^\dagger_\alpha(\vec{p}) \right) \tag{5.87}$$

we obtain anti-commutation relations for the Fourier modes

$$\begin{aligned} \{u^a_\gamma(\vec{p}) b_\gamma(\vec{p}), \bar{u}_{\gamma,b}(\vec{q}) b^\dagger_\gamma(\vec{q})\} &= (p\cdot\gamma + m)^a{}_b\, 2e(\vec{p})(2\pi)^3\delta^3(\vec{p} - \vec{q}). \\ \{\bar{v}_{\gamma,b}(\vec{p}) a_\gamma(\vec{p}), v^a_\gamma(\vec{q}) a^\dagger\gamma(\vec{q})\} &= (p\cdot\gamma - m)^a{}_b\, 2e(\vec{p})(2\pi)^3\delta^3(\vec{p} - \vec{q}). \end{aligned} \tag{5.88}$$

It is convenient to split these relations into contributions from quantum operators and contributions from spinor solutions. We postulate simple anti-commutation relations for the creation and annihilation operators

$$\{a_\alpha(\vec{p}), a^\dagger_\beta(\vec{q})\} = \{b_\alpha(\vec{p}), b^\dagger_\beta(\vec{q})\} = \delta_{\alpha\beta}\, 2e(\vec{p})(2\pi)^3\delta^3(\vec{p} - \vec{q}). \tag{5.89}$$

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[18] The correct normalisation can be derived from $\dot{F} = -\{H, F\}$.

[19] Although the Poisson brackets are anti-symmetric in most cases, they are symmetric for two Grassmann odd elements.

[20] The sign is determined by the relation $[H, \psi^a(\vec{x})] = -i\dot{\psi}^a(\vec{x})$.

Together with the above anti-commutators, they imply the completeness relation for the basis of spinor solutions

$$u_\alpha(\vec{p})\bar{u}_\alpha(\vec{p}) = p{\cdot}\gamma + m,$$
$$v_\alpha(\vec{p})\bar{v}_\alpha(\vec{p}) = p{\cdot}\gamma - m. \tag{5.90}$$

In the Weyl representation these relations are reproduced by the particular choice for the spinors u, v.

$$u_\alpha(\vec{p}) = \begin{pmatrix} (p{\cdot}\sigma)^{1/2}\xi_\alpha \\ (p{\cdot}\bar{\sigma})^{1/2}\xi_\alpha \end{pmatrix}, \qquad v_\alpha(\vec{p}) = \begin{pmatrix} (p{\cdot}\sigma)^{1/2}\xi_\alpha \\ -(p{\cdot}\bar{\sigma})^{1/2}\xi_\alpha \end{pmatrix}, \tag{5.91}$$

where ξ_α is an orthonormal basis of 2-spinors.

Dirac Sea. The Pauli exclusion principle for fermions states that each state can be occupied only once. It follows from the (not explicitly written) anti-commutators

$$\{a^\dagger_\alpha(\vec{p}), a^\dagger_\beta(\vec{q})\} = \{b^\dagger_\alpha(\vec{p}), b^\dagger_\beta(\vec{q})\} = 0 \tag{5.92}$$

that

$$(a^\dagger_\alpha(\vec{p}))^2 = (b^\dagger_\alpha(\vec{p}))^2 = 0. \tag{5.93}$$

Dirac used the exclusion principle to make useful proposals concerning negative-energy states in relativistic quantum mechanics and the prediction of anti-particles.

The Dirac equation has positive and negative solutions. Furthermore the solutions carry an (electrical) charge. Dirac proposed that all negative-energy states are already occupied in the vacuum and cannot be excited further. This picture is called the Dirac sea, and it explained how to avoid negative-energy solutions.

Continuing this thought, there is now the option to remove an excitation from one of the occupied states. This hole state would not only have positive energy, but also carry charges exactly opposite to the ones of the regular positive-energy solutions. In this way he predicted the existence of positrons as the anti-particles of electrons. The prediction was soon thereafter confirmed in experiment.

Our view of QFT today is different, so let us compare:

- Positive-energy solutions of ψ are associated to $a^\dagger$.
- Negative-energy solutions of ψ are associated to b.
- Let us define $c^\dagger = b$, it is our choice.
- The vacuum is annihilated by $c^\dagger$. All c-states are occupied.
- A hole in the Dirac sea $c = b^\dagger$ creates an anti-particle.

QFT for Dirac particles works as predicted, but:

- There is no need for a Dirac sea.
- Negative-energy solutions are defined as annihilation operators, not as creation operators with occupied states.
- Dirac's argument relies on the exclusion principle, it works for fermions only. QFT can also deal with bosons.

- The Dirac equation has real solutions (see later) just as well as the Klein–Gordon equation has complex solutions.
  The existence of anti-particles is unrelated to spin and the Dirac equation. It is a consequence of CPT.

Correlators and Propagators. We now have all we need to compute correlators of the free quantum fields. There are two non-vanishing correlators of two fields

$$\begin{aligned} \Delta^{\mathrm{D}a}_{+}{}_{b}(x-y) &= \langle 0|\psi^a(x)\bar\psi_b(y)|0\rangle, \\ \Delta^{\mathrm{D}a}_{-}{}_{b}(x-y) &= \langle 0|\bar\psi_b(y)\psi^a(x)|0\rangle. \end{aligned} \tag{5.94}$$

They can be expressed in terms of the correlator of two scalars Δ_+ with an additional operator

$$\begin{aligned} \Delta^{\mathrm{D}a}_{+}{}_{b}(x) &= \int \frac{d^3\vec p\, e^{-ip\cdot x}}{(2\pi)^3\, 2e(\vec p)}\, (p\cdot\gamma+m)^a{}_b = (i\gamma\cdot\partial+m)^a{}_b\Delta_+(x), \\ \Delta^{\mathrm{D}a}_{-}{}_{b}(-x) &= \int \frac{d^3\vec p\, e^{-ip\cdot x}}{(2\pi)^3\, 2e(\vec p)}\, (p\cdot\gamma-m)^a{}_b = (i\gamma\cdot\partial-m)^a{}_b\Delta_+(x). \end{aligned} \tag{5.95}$$

When acting with the Dirac equation on the correlator, it combines with the operator to give the Klein–Gordon equation acting on Δ_+, e.g.

$$\begin{aligned} &\big(i(\partial/\partial x^\mu)\gamma^\mu - m\big)^a{}_b\Delta^{\mathrm{D}b}_{+}{}_{c}(x) \\ &= (i\gamma\cdot\partial-m)^a{}_b(i\gamma\cdot\partial+m)^b{}_c\Delta_+(x) \\ &= \delta^a_c\big((\partial/\partial x)^2 - m^2\big)\Delta_+(x) = 0. \end{aligned} \tag{5.96}$$

Likewise, the unequal time anti-commutator

$$\{\psi^a(x),\bar\psi_b(y)\} = \Delta^{\mathrm{D}a}{}_b(y-x) \tag{5.97}$$

can be written in terms of the one for the scalar field

$$\Delta^{\mathrm{D}}(x) = (i\gamma\cdot\partial - m)\Delta(x). \tag{5.98}$$

As such it satisfies the Dirac equation and vanishes for space-like separations[21]

$$\{\psi^a(x),\bar\psi_b(y)\} = 0 \quad \text{for } (x-y)^2 > 0. \tag{5.99}$$

For the Dirac equation with a source, the same methods we introduced earlier for the scalar field apply. The propagator is a spinor matrix and defined via the equations

$$\begin{aligned} (-i\gamma\cdot\partial+m)^a{}_b G^{\mathrm{D}b}{}_c(x) &= \delta^a_c\delta^4(x), \\ G^{\mathrm{D}a}{}_b(x)(-i\gamma\cdot\partial^\dagger+m)^b{}_c &= \delta^a_c\delta^4(x), \end{aligned} \tag{5.100}$$

[21]The fact that an anti-commutator vanishes is not in contradiction with causality. Typically we can observe only fermion bilinears which are bosonic and do commute.

supplemented by suitable boundary conditions. By the same reasons as above, we can express the Dirac propagator through the scalar propagator

$$G^{\mathrm{D}a}{}_b(x) = (i\gamma\cdot\partial + m)^a{}_b G(x). \tag{5.101}$$

Obviously, one has the same relations as before, e.g. for the retarded propagator

$$G^{\mathrm{D}}_{\mathrm{R}}(x) = i\theta(t)\varDelta^{\mathrm{D}}(x). \tag{5.102}$$

There are some other useful relationships between correlators and propagators in momentum space which are worth emphasising because they hold generally.

First of all, by construction the propagator is the inverse of the kinetic term in the action[22]

$$G^{\mathrm{D}}(p) = (-\gamma\cdot p + m)^{-1} = \frac{\gamma\cdot p + m}{p^2 + m^2} \tag{5.103}$$

The corresponding correlators and unequal time commutators take the form

$$\begin{aligned}
\varDelta^{\mathrm{D}}_{\pm}(p) &= \pm 2\pi\delta(p^2 + m^2)\theta(\pm p_0)(p\cdot\gamma + m),\\
\varDelta^{\mathrm{D}}(p) &= 2\pi\delta(p^2 + m^2)\,\mathrm{sign}(p_0)(p\cdot\gamma + m).
\end{aligned} \tag{5.104}$$

These reflect precisely the residues times a delta function localised at the position of the pole along with some restriction to positive or negative energies.

The construction of the propagator and its relationship to correlators and commutators can be used as a shortcut in deriving the latter. Large parts of the canonical quantisation procedure can thus be avoided in practice.

5.7 Complex Field

The Dirac spinor is complex and the Lagrangian has the obvious U(1) global symmetry

$$\psi'(x) = e^{i\alpha}\psi, \qquad \bar\psi'(x) = e^{-i\alpha}\bar\psi. \tag{5.105}$$

The symmetry has a corresponding conserved Noether current[23]

$$J^\mu = \frac{\delta\psi^a}{\delta\alpha}\,\frac{\delta\mathcal{L}}{\delta\partial_\mu\psi^a} = -\bar\psi\gamma^\mu\psi. \tag{5.106}$$

The time component of the current was used earlier to define a positive definite probability density, $-J^0 = \psi^\dagger\psi$. However, if one follows the spin-statistics theorem and let ψ be Grassmann odd, the density is not positive. In particular, it changes sign for the charge conjugate solution

$$J^\mu_{\mathrm{C}} = -\bar\psi_{\mathrm{C}}\gamma^\mu\psi_{\mathrm{C}} = \bar\psi\gamma^\mu\psi = -J^\mu. \tag{5.107}$$

[22]The poles should be shifted away from the real axis to accommodate for the desired boundary conditions.

[23]Note the order of terms.

Nevertheless, the current is conserved and defines a conserved Noether charge

$$Q = \int d^3\vec{x}\, J^0 = -\int d^3\vec{x}\, \psi^\dagger\psi. \tag{5.108}$$

It leads to the usual charge assignments for the complex field

$$[Q, \psi(x)] = \psi(x), \qquad [Q, \bar{\psi}(x)] = -\bar{\psi}(x). \tag{5.109}$$

5.8 Real Field

The Dirac field has four independent particle modes, $a^\dagger_\alpha(\vec{p})$ and $b^\dagger_\alpha(\vec{p})$, for each three-momentum. From the classification of Poincaré UIR's we know that the irreducible representation for spin $j = \frac{1}{2}$ has only two spin orientations for each three-momentum.

Reality Condition. This discrepancy is associated to the existence of charge conjugate solutions. We can remove the additional solutions by imposing a reality condition on ψ, namely[24]

$$\psi_\mathrm{C} = \psi. \tag{5.110}$$

A spinor which satisfies this condition is called a Majorana spinor. There exist representations of the Clifford algebra where all γ^μ are purely imaginary. In this basis the Dirac equation is real, and it makes sense to restrict ψ to real (Grassmann odd) numbers.

For the momentum modes we can use the identity

$$u_\alpha(\vec{p}) = \gamma_\mathrm{C} v^*_{\alpha'}(\vec{p}), \tag{5.111}$$

to show that the identification $\psi_\mathrm{C} = \psi$ implies

$$a_\alpha(\vec{p}) = b_{\alpha'}(\vec{p}) \tag{5.112}$$

The identification may involve some translation between the bases a_α and $b_{\alpha'}$. It reduces the modes of the Dirac field by a factor of two.

2-Spinors. Let us consider a real spinor $\psi = (\psi_\mathrm{L}, \psi_\mathrm{R})$ in the Weyl representation. The reality condition implies

$$\psi_\mathrm{L} = -i\sigma^2\psi_\mathrm{R}^{\dagger\mathrm{T}} = \frac{1}{\sqrt{2}}\,\chi. \tag{5.113}$$

This allows to write the Lagrangian in terms of the 2-spinor field χ as[25]

$$\mathcal{L} = \chi^\dagger i\bar{\sigma}\cdot\partial\chi + \tfrac{i}{2}m\chi^\mathrm{T}\sigma^2\chi - \tfrac{i}{2}m\chi^\dagger\sigma^2\chi^{\dagger\mathrm{T}}. \tag{5.114}$$

The Lagrangian for the Dirac field can be written as two identical copies of this.[26]

[24] One could also use any other complex phase $e^{i\alpha}$ between ψ_C and ψ.

[25] The two mass terms are anti-symmetric in χ, which requires the classical field χ to be an odd Grassmann number.

[26] The U(1) global symmetry of the Dirac equation is recovered as a SO(2) rotation symmetry of the two fields χ.

Parity. Note that parity interchanges ψ_L and ψ_R. The reality condition relates the two, hence

$$\chi'(t,-\vec{x}) = -i\sigma^2\chi^{\dagger\mathrm{T}}(t,\vec{x}). \tag{5.115}$$

As this transformation also sends the field χ to its complex conjugate $\chi^{\dagger\mathrm{T}}$, it is usually viewed as the combination CP of charge conjugation C and parity P

$$CP\chi(t,\vec{x})(CP)^{-1} = -i\sigma^2\chi^{\dagger\mathrm{T}}(t,-\vec{x}). \tag{5.116}$$

In that sense there cannot be individual C and P transformations and only CP can be a symmetry.

An alternative point of view is that C was used to define the reality condition. Hence C is preserved by construction, and the parity operation P is well-defined on its own.

Technically, both points of view have the same content: They merely use the same words to refer to different operators. They are related by identifying $C' = 1$ and $P' = CP$, where the primed operations refer to the latter approach.

5.9 Massless Real Field

So far we have assumed a non-zero mass m. Let us now consider the massless case which has some special features. We will assume a real (Majorana) field.

First, let us compare to the UIR's of the Poincaré group: There are two particles $a^\dagger_\alpha(\vec{p})$ for each momentum. Conversely, a UIR with fixed helicity has merely one state for each momentum. The two particles correspond to UIR's with helicity $h = \pm\frac{1}{2}$. In fact, helicity states must always come in pairs in QFT. One cannot construct a real Lagrangian which describes just one helicity.

Interestingly, the splitting of representations leads to an enhancement of symmetry. For $m = 0$, the Lagrangian in terms of 2-spinors reads

$$\mathcal{L} = \chi^\dagger i\bar{\sigma}{\cdot}\partial\chi. \tag{5.117}$$

Quite obviously, this Lagrangian has a global U(1) symmetry

$$\chi' = e^{i\alpha}\chi. \tag{5.118}$$

It is called chiral symmetry. The associated Noether current reads

$$J^\mu = -\chi^\dagger\bar{\sigma}^\mu\chi. \tag{5.119}$$

At the level of 4-spinors chiral symmetry is represented by the transformation

$$\psi' = \exp(-i\alpha\gamma^5)\psi. \tag{5.120}$$

The adjoint spinor transforms with the same factor

$$\bar{\psi}' = \bar{\psi}\exp(-i\alpha\gamma^5). \tag{5.121}$$

This is transformed to the inverse factor by a single gamma matrix

$$\exp(-i\alpha\gamma^5)\gamma^\mu = \gamma^\mu \exp(i\alpha\gamma^5). \tag{5.122}$$

The massless Lagrangian is therefore invariant under chiral transformations. Here the conserved current is the so-called axial vector current

$$J^\mu = -\bar{\psi}\gamma^5\gamma^\mu\psi. \tag{5.123}$$

5.10 Chiral Transformations and Masses

Let us consider the above chiral transformations in the presence of mass

$$\begin{aligned}\mathcal{L}' &= \bar{\psi}\big(i\gamma{\cdot}\partial - m\exp(2i\alpha\gamma^5)\big)\psi \\ &= \bar{\psi}\big(i\gamma{\cdot}\partial - m\cos(2\alpha) - m\sin(2\alpha)i\gamma^5\big)\psi.\end{aligned} \tag{5.124}$$

On the one hand, it shows that masses break chiral symmetry. On the other hand, we have learned that there are two types of mass terms

$$\bar{\psi}\psi \quad \text{and} \quad i\bar{\psi}\gamma^5\psi. \tag{5.125}$$

They are both equivalent under a chiral transformation. When they appear simultaneously, the physical squared mass is the sum of the squares of the two coefficients.

For a real field, the transformed Lagrangian in terms of 2-spinors reads

$$\mathcal{L}' = \chi^\dagger i\bar{\sigma}{\cdot}\partial\chi + \tfrac{i}{2}me^{2i\alpha}\chi^{\mathrm{T}}\sigma^2\chi - \tfrac{i}{2}me^{-2i\alpha}\chi^\dagger\sigma^2\chi^{\dagger\mathrm{T}}. \tag{5.126}$$

We could thus also introduce a complex m such that the physically relevant squared mass is just $m\bar{m} = |m|^2$.