Quantum Field Theory I — Problem Set 1

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1. Classical particle in an electromagnetic field

Consider the classical Lagrangian density of a particle of mass m and charge q, moving in an electromagnetic field, specified by the electric potential $\phi(x)$ and the magnetic vector potential $A_i(x)$:

$$\mathcal{L} = \frac{m}{2}(\partial_t x^i)^2 + qA_i(x)\partial_t x^i - q\phi(x). \tag{1}$$

Find

a) The canonical momentum conjugate to the coordinate x_i.

b) The equations of motion corresponding to the Lagrangian density.

c) The Hamiltonian of the system

Compare your results to a free particle.

2. Stress-energy tensor

Consider the variational principle:

$$\delta S = 0 = \delta \int d^4x \, \mathcal{L}(\phi, \pi). \tag{2}$$

The Lagrangian density $\mathcal{L}$ is a function of the two classical fields $\phi(x)$ and $\pi_\mu(x) = \partial_\mu \phi(x)$. Note that $\mathcal{L}$ does not depend directly on the space-time coordinate x^μ, but only indirectly through ϕ and π. Show that the conserved Noether current associated with infinitesimal space-time translations

$$x^\mu \to x^\mu + \epsilon^\mu \tag{3}$$

is the stress-energy tensor $T^{\mu\nu}$ given by

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \pi_\mu} \partial^\nu \phi - g^{\mu\nu} \mathcal{L}. \tag{4}$$

Remind yourself how a general function $f(x^\mu)$ of the space-time coordinates will transform under an infinitesimal translation.
Note that x^μ is a standard Minkowski-space coordinate, so that x^0 is the time.
$\eta^{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$ is the metric tensor.

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3. Coherent quantum oscillator

Consider the Hamiltonian of a quantum harmonic oscillator:

$$\mathcal{H} = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \,. \tag{5}$$

a) Introduce ladder operators to diagonalise the Hamiltonian.

b) Calculate the expectation values of the number operator $N \sim a^\dagger a$ as well as of the x and p operator in a general number state $|n\rangle$.

c) Calculate the variances Δx, Δp and ΔN in the same state $|n\rangle$ and use them to determine the Heisenberg uncertainty of $|n\rangle$.

d) Show that the coherent state

$$|\alpha\rangle = e^{\alpha p}|0\rangle \tag{6}$$

is an eigenstate of the annihilation operator you defined in a).

e) Calculate the time-dependent expectation values of x, p and N:

$$\langle\alpha|x(t)|\alpha\rangle \tag{7}$$

$$\langle\alpha|p(t)|\alpha\rangle \tag{8}$$

$$\langle\alpha|N(t)|\alpha\rangle \tag{9}$$

as well as the corresponding variances to determine the uncertainty of the state $|\alpha\rangle$. Compare your result with the result obtained in c).

4. Relativistic point particle

The action of a relativistic point particle is given by

$$S = -\alpha \int_{\mathcal{P}} ds \tag{10}$$

with the relativistic line element

$$ds^2 = -g_{\mu\nu} dX^\mu dX^\nu = dt^2 - dx^2 - dy^2 - dz^2 \tag{11}$$

and α a (yet to be determined) constant.
The path $\mathcal{P}$ between two points X_1^μ and X_2^μ can be parametrised by a parameter τ. With that, the integral of the line element ds becomes an integral over the parameter

$$S = -\alpha \int_{\tau_1}^{\tau_2} d\tau \sqrt{-g_{\mu\nu} \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \tau}} \,. \tag{12}$$

a) Parametrise the path by the time coordinate t (i.e. x^0) and take the non-relativistic limit $|\partial_0 x^\mu| \ll 1$ to determine the value of the constant α.

b) Derive the equations of motion by varying the action. *Hint:* You may want to determine the canonically conjugate momentum first.