Quantum Field Theory I — Problem Set 7

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1. Polarisation vectors of a massless vector field

Each Fourier mode in the plane wave expansion of a massless vector field has the form

$$A_\mu^{(\lambda)}(\vec{p};x) = N(\vec{p})\,\epsilon_\mu^{(\lambda)}(\vec{p})\,e^{ip\cdot x} \tag{1}$$

Without any loss of generality the polarisation vectors $\epsilon_\mu^{(\lambda)}(\vec{p})$ can be chosen to form a four-dimensional orthonormal system satisfying

$$\epsilon_\mu^{(\lambda)}(\vec{p})\,\epsilon^{(\kappa)\mu}(\vec{p}) = \eta^{\lambda\kappa}. \tag{2}$$

a) Show that the following choice satisfies (2)

$$\epsilon_\mu^{(0)}(\vec{p}) = n_\mu, \tag{3}$$

$$\epsilon_\mu^{(1)}(\vec{p}) = (0, \vec{\epsilon}^{\,(1)}(\vec{p})), \tag{4}$$

$$\epsilon_\mu^{(2)}(\vec{p}) = (0, \vec{\epsilon}^{\,(2)}(\vec{p})), \tag{5}$$

$$\epsilon_\mu^{(3)}(\vec{p}) = \left(p_\mu + n_\mu(p\cdot n)\right)/|p\cdot n|, \tag{6}$$

where $n_\mu = (1,0)$ and $\vec{p}\cdot\vec{\epsilon}^{\,(k)}(\vec{p}) = 0$ as well as $\vec{\epsilon}^{\,(k)}(\vec{p})\cdot\vec{\epsilon}^{\,(l)}(\vec{p}) = \delta^{kl}$.

b) Use the polarisation vectors to verify the completeness relation

$$\sum_{\lambda=0}^{3} \eta_{\lambda\lambda}\,\epsilon_\mu^{(\lambda)}(\vec{p})\,\epsilon_\nu^{(\lambda)}(\vec{p}) = \eta_{\mu\nu}. \tag{7}$$

c) Show for the physical modes of the photon that

$$\sum_{\lambda=1}^{2} \epsilon_\mu^{(\lambda)}(\vec{p})\,\epsilon_\nu^{(\lambda)}(\vec{p}) = \eta_{\mu\nu} - \frac{p_\mu p_\nu}{(p\cdot n)^2} - \frac{p_\mu n_\nu + p_\nu n_\mu}{p\cdot n}\,. \tag{8}$$

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2. Spinor helicity framework

The spinor helicity framework is a method to conveniently work with massless particles and their helicity modes.
Write a momentum 4-vector p_μ as a 2×2 matrix P

$$P = \sigma^\mu p_\mu. \tag{9}$$

a) Show that the inverse transformation is given by $p_\mu = -\frac{1}{2}\operatorname{tr}(\bar\sigma_\mu P)$.

b) Show that $\det P = -p^2$.

c) Explain why the momentum P of a massless particle can be expressed as a product of a (bosonic) 2-spinor λ and its hermitian conjugate $\lambda^\dagger$

$$P = \lambda\lambda^\dagger. \tag{10}$$

Is λ uniquely determined through p? What can you say about the energy p_0?

d) Show that the Lorentz-invariant integral over the light cone can be expressed as a plain integral over all λ's

$$\int \frac{dp_1\, dp_2\, dp_3}{(2\pi)^3\, 2e(\vec p)} f(\vec p) = \int \frac{d\lambda_1 d\lambda_1^* \, d\lambda_2 d\lambda_2^*}{4(2\pi)^4}\, f(\vec p(\lambda, \lambda^\dagger)). \tag{11}$$

Hint: As a fourth variable for the integral on the l.h.s. you may use the undetermined complex phase $\varphi = -\frac{i}{2}\log(\lambda_1/\lambda_1^*)$ of λ_1 integrated over $0 \le \varphi < 2\pi$.

Given some non-trivial 2-spinor μ (not proportional to λ), two polarisation vectors with helicity $h = \pm 1$ can be constructed as

$$\epsilon_\mu^{(+)}(\vec p) = \frac{\mu^\dagger \bar\sigma_\mu \lambda}{\mu^\dagger \sigma^2 \lambda^*}\,, \qquad \epsilon_\mu^{(-)}(\vec p) = \frac{\lambda^\dagger \bar\sigma_\mu \mu}{\lambda^\intercal \sigma^2 \mu}\,. \tag{12}$$

e) Show that $p \cdot \varepsilon^{(\pm)}(\vec p) = 0$.

f) Show that a change in μ acts as a gauge transformation on the polarisation vectors, i.e. $\delta\varepsilon_\mu^{(\pm)} \sim p_\mu$. *Hint:* Parametrise $\delta\mu$ as a linear combination of μ and λ.

3. The photon propagator with a gauge fixing term

Consider the Lagrangian for a free massless vector field modified by a gauge fixing term

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}\xi(\partial_\lambda A^\lambda)^2. \tag{13}$$

a) Show that the Lagrangian is equivalent to the following one up to a total derivative

$$\mathcal{L}' = -\tfrac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu - \tfrac{1}{2}(\xi - 1)(\partial_\lambda A^\lambda)^2. \tag{14}$$

b) Show that the equal-time commutation relations are (you may use $\mathcal{L}$ or $\mathcal{L}'$)

$$\big[A^\mu(t,\vec x), A^\nu(t,\vec y)\big] = 0, \tag{15}$$

$$\big[A^\mu(t,\vec x), \dot A^\nu(t,\vec y)\big] = i\left(\eta^{\mu\nu} + \frac{\xi - 1}{\xi}\,\delta_0^\mu\delta_0^\nu\right)\delta^3(\vec x - \vec y), \tag{16}$$

$$\big[\dot A^\mu(t,\vec x), \dot A^\nu(t,\vec y)\big] = -i\,\frac{\xi - 1}{\xi}\,\big(\delta_0^\mu\delta^{\nu k} + \delta_0^\nu\delta^{\mu k}\big)\,\partial_k\delta^3(\vec x - \vec y). \tag{17}$$

c) Show that the propagator is given by

$$G^{\mu\nu}(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot x}\, \frac{1}{p^2}\left(\eta^{\mu\nu} + \frac{\xi - 1}{\xi}\,\frac{p^\mu p^\nu}{p^2}\right). \tag{18}$$