Quantum Field Theory I — Problem Set 9

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1. 4-point interaction in scalar QED

Consider a $U(1)$ gauge theory with two complex massive scalar fields ϕ, χ and one vector field A_μ. Each of the scalar fields is coupled to the gauge field and they both have the same coupling e. The Lagrangian density of the theory is given by

$$\mathcal{L} = -\tfrac{1}{2}(D_\mu\phi)^\dagger D^\mu\phi - \tfrac{1}{2}(D_\mu\chi)^\dagger D^\mu\chi - \tfrac{1}{2}m^2\phi^\dagger\phi - \tfrac{1}{2}m^2\chi^\dagger\chi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{1}$$

with the covariant derivative

$$D_\mu = \partial_\mu + ieA_\mu(x) \tag{2}$$

and $F^{\mu\nu}$, the electro-magnetic field strength tensor. Use the Feynman gauge fixing term.

In this exercise we are interested in obtaining the time-ordered 4-point correlation function for two fields of type ϕ and two fields of type χ. To be precise we want to compute the first interaction term of ϕ with χ in the expansion in the perturbative parameter e.

a) First find the interaction Hamiltonian H_{int} of the theory.

b) Perform an expansion of the time ordered 4-point correlation function in e up to the first term that allows for an interaction with the gauge field A_μ

$$\begin{aligned}&\langle 0|\mathrm{T}\left\{\phi(x_1)\phi^\dagger(x_2)\chi(x_3)\chi^\dagger(x_4)\right\}|0\rangle_{\text{int}}\\ &= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\mathrm{T}\left\{\phi(x_1)\phi^\dagger(x_2)\chi(x_3)\chi^\dagger(x_4)\exp\left[-i\int_{-T}^{T}dt\,H_{\text{int}}(t)\right]\right\}|0\rangle}{\langle 0|\mathrm{T}\left\{\exp\left[-i\int_{-T}^{T}dt\,H_{\text{int}}(t)\right]\right\}|0\rangle}.\end{aligned} \tag{3}$$

Hint: You may discard the terms which do not contribute to the correlator.

c) Make use of Wick's theorem to contract the fields in the interaction term you obtained in problem b).

d) Focus on the contribution(s) where ϕ and χ interact non-trivially: Insert the Fourier transformed propagators of the scalar and vector fields into your result

$$\begin{aligned}G_{\mathrm{F}}(x-y) = i\langle 0|\mathrm{T}\{\phi^\dagger(x)\phi(y)\}|0\rangle &= \int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x-y)}}{p^2+m^2-i\epsilon},\\ G_{\mathrm{F}}^{\mu\nu}(x-y) = i\langle 0|\mathrm{T}\{A_\mu(x)A_\nu(y)\}|0\rangle &= \int \frac{d^4p}{(2\pi)^4}\frac{\eta^{\mu\nu}e^{-ip(x-y)}}{p^2-i\epsilon}.\end{aligned} \tag{4}$$

Simplify your result by performing the integration over the internal spatial variables. How can you interpret the individual factors in your result?

e) How can you interpret the terms that do not lead to an interaction of ϕ and χ? Can you find a diagrammatic representation of those terms? How do you interpret the limit of $T \to \infty$ in equation (3)?

f) *Optional:* How will your result in d) change if you use a different gauge? E.g. use a Lorentz gauge fixing term with $\xi \neq 1$. *Hint:* The gauge affects only $G_{\mathrm{F}}^{\mu\nu}$; try partial fractions to simplify the result.