Quantum Field Theory I — Problem Set 12

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1. Volume of higher-dimensional spheres

The integrands of D-dimensional loop integrals often are spherically symmetric functions $F(\vec{x}) = F(|\vec{x}|)$ (or they can be brought into this form, see Problem 2). The angular part of the integral in spherical coordinates yields the volume of the $(D-1)$-dimensional sphere S^{D-1}

$$\int d^D\vec{x}\, F(|\vec{x}|) = \mathrm{Vol}(S^{D-1}) \int_0^\infty r^{D-1} dr\, F(r). \tag{1}$$

In particular, in view of the dimensional regularisation scheme, where D is assumed to be a real number, we need a suitable formula for the volume as an analytic function of D. Use the well-known result

$$\int_{-\infty}^{\infty} dx\, \exp(-x^2) = \sqrt{\pi}, \tag{2}$$

to show that the volume of the $(D-1)$-sphere is

$$\mathrm{Vol}(S^{D-1}) = \frac{2\pi^{D/2}}{\Gamma(D/2)}\,. \tag{3}$$

2. Feynman and Schwinger parameters

a) To evaluate loop diagrams one combines propagators with the use of *Feynman parameters*. The basic version is

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[xA + (1-x)B]^2}, \tag{4}$$

but it can be generalised to n propagators elevated to some arbitrary power

$$\frac{1}{\prod_{i=1}^n A_i^{\nu_i}} = \frac{\Gamma\left(\sum_{i=1}^n \nu_i\right)}{\prod_{i=1}^n \Gamma(\nu_i)} \int_0^1 \left(\prod_{i=1}^n dx_i\right) \delta\left(1 - \sum_{i=1}^n x_i\right) \frac{\prod_{i=1}^n x_i^{\nu_i - 1}}{\left[\sum_{i=1}^n x_i A_i\right]^{\sum_{i=1}^n \nu_i}}. \tag{5}$$

Prove (5) recursively.

b) Another useful parametrisation is the *Schwinger parametrisation*:

$$\frac{1}{A^\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty d\alpha\, \alpha^{\nu-1} e^{-\alpha A}. \tag{6}$$

Prove (6).

3. Electron self energy structure

In QED, the electron two-point function $F(p,q) = -i(2\pi)^4\delta^4(p+q)M(p)$ receives contributions from self energy diagrams.

a) Draw the Feynman diagrams corresponding to the one- and two-loop contributions. Which of these diagrams are one-particle irreducible?

b) For the one-loop case, write down the expression for $M(p)$ using the massive QED Feynman rules in momentum space and argue why the integral is divergent.

c) Explain why one can make the ansatz

$$M = p\cdot\gamma\, M_{\mathrm{V}} + m\, M_{\mathrm{S}}, \tag{7}$$

where $M_{\mathrm{V,S}}$ are scalar functions. Write down integral expressions for them.