0   Overview

Quantum Field Theory is the quantum theory of fields just like Quantum Mechanics describes quantum particles. Here, a the term "field" refers to one of the following:

- A field of a classical field theory, such as Electromagnetism.
- A wave function of a particle in Quantum Mechanics. This is why QFT is sometimes called "Second Quantisation".
- A smooth approximation to some property in a solid, e.g. the displacement of atoms in a lattice.
- Some function of space and time describing some physics.

Usually, excitations of the quantum field will be described by "particles". In QFT the number of these particles is not conserved, they are created and annihilated when they interact. It is precisely what we observe in elementary particle physics, hence QFT has become the mathematical framework for this discipline.

This lecture series gives an introduction to the basics of quantum field theory. It describes how to quantise the basic types of (relativistic) fields, how to handle their quantum operators and how to treat (sufficiently weak) interactions. Furthermore, we discuss symmetries, infinities and running couplings. The goal of the course is a derivation of particle scattering processes in basic QFT models.

The continuation of this lecture course, QFT II, introduces an alternative quantisation framework, the path integral.[1] It is applied towards formulating the Standard Model of Particle Physics by means of non-abelian gauge theory and spontaneous symmetry breaking.

What Else is QFT?   Many points of view.

After attending this course, you may claim QFT is all about another 1000 ways to treat free particles and harmonic oscillators. True, some of the few systems we can solve exactly in theoretical physics; everything else is approximation. Physics, not maths!

Look more carefully: QFT is a very rich subject, can learn about many things, some of which have attained a mythological status:

- anti-particles, anti-matter,
- vacuum energy,
- tachyons,
- ghosts,

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[1] Path integral much more convenient to use than canonical quantisation discussed here. However, some important concepts are not as obvious as in canonical quantisation, e.g. notion of particles, scattering and importantly unitarity.

- infinities,
- mathematically inconsistent (?).

Infinities. How to deal with infinities?

Quote Dirac about QED (1975): "I must say that I am very dissatisfied with the situation, because this so-called 'good theory' does involve neglecting infinities which appear in its equations, neglecting them in an arbitrary way. This is just not sensible mathematics. Sensible mathematics involves neglecting a quantity when it is small — not neglecting it just because it is infinitely great and you do not want it!"

Almost true, but it's neither neglecting nor in an arbitrary way.

Infinities are one reason why QFT is claimed to be mathematically ill-defined or even inconsistent. Yet QFT is a well-defined and consistent calculational framework to predict certain particle observables to extremely high precision.

Many points of view, one is that it's our own fault: QFT is somewhat idealised, assumes infinitely extended fields (IR) with infinite spatial resolution (UV);[2] no wonder that the theory produces infinities. Still, better to stick to idealised assumptions and live with infinities than use enormous discrete systems (actual solid state systems).

A physics reason why these infinities can be absorbed somehow: Our observable everyday physics should neither depend on how large the universe actually is (IR) nor on how smooth or coarse-grained space is (UV).

Can use infinities to learn about allowable particle interactions. Leads to curious effects: running coupling and quantum anomalies.

More later (towards the end of the semester).

Uniqueness. A related issue is uniqueness of the formulation. Alike QM, QFT does not have a unique or universal formulation.

For instance, many meaningful things in QM/QFT are actually equivalence classes of objects. More convenient to work with specific representatives of these classes. Have to bear in mind that only equivalence class is meaningful, hence many ways to describe the same physical object.

Usage of classes goes further, not just classes objects. Often have to consider classes of models rather than specific models. Something we have to accept, something QFT forces upon us.

We'll notice that QFT does what it wants, not necessarily what we want. Cannot expect to get what we want using bare input parameters. Different formulations of the same model naively may give different results. Must learn to adjust the input to the desired output, then find agreement. Just have to make sure that there is more output than input, otherwise QFT would be nice but meaningless exercise.

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[2]Two sources for infinities: UV and IR.

Nice feature: Can hide infinities there in a self-consistent way.

Enough. Just some words of warning: Must give up some views on physics you have become used to. Only then you can understand something new.[3] Let's start with something concrete. Discuss the tricky issues when they arise.

Important Concepts.

- unitarity – probabilistic framework.
- locality – interactions are strictly local.
- causality – special relativity.
- symmetries – exciting algebra and geometry.
- analyticity – complex analysis.

0.1   Prerequisites

- Classical mechanics (brief review in first lecture)
- Quantum mechanics (brief review in first lecture)
- Electrodynamics (as a simple classical field theory)
- Mathematical methods in physics (HO, Fourier transforms, . . .)

0.2   Contents

1.	Classical and Quantum Mechanics	(2 lectures)
2.	Classical Free Scalar Field	(3 lectures)
3.	Scalar Field Quantisation	(5 lectures)
4.	Symmetries	(5 lectures)
5.	Free Spinor Field	(8 lectures)
6.	Free Vector Field	(6 lectures)
7.	Interactions	(4 lectures)
8.	Correlation Functions	(6 lectures)
9.	Particle Scattering	(4 lectures)
10.	Scattering Matrix	(5 lectures)
11.	Loop Corrections	(5 lectures)

Indicated are the approximate number of 45-minute lectures. Altogether, the course consists of 54 lectures which includes one overview lecture.

0.3   References

There are many text books and lecture notes on quantum field theory. Here is a selection of well-known ones:

- M. E. Peskin, D. V. Schroeder, "An Introduction to Quantum Field Theory", Westview Press (1995)

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[3]E.g. in special relativity: space and time become space time.

- C. Itzykson, J.-B. Zuber, "Quantum Field Theory", McGraw-Hill (1980)
- P. Ramond, "Field Theory: A Modern Primer", Westview Press (1990)
- M. Srednicki, "Quantum Field Theory", Cambridge University Press (2007)
- M. Kaku, "Quantum Field Theory", Oxford University Press (1993)
- online: D. Tong, "Quantum Field Theory", lecture notes,
  `http://www.damtp.cam.ac.uk/user/tong/qft.html`
- online: M. Gaberdiel, "Quantenfeldtheorie", lecture notes (in German),
  `http://www.itp.phys.ethz.ch/people/gaberdim`
- ...

Peskin & Schroeder may be closest to this lecture course.[4]

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[4]We will not follow Peskin & Schroeder literally.

1   Classical and Quantum Mechanics

To familiarise ourselves with the basics, let us review some elements of classical and quantum mechanics. Then we shall discuss some problems of combining quantum mechanics with special relativity.

1.1   Classical Mechanics

Consider a classical non-relativistic particle in a potential. Described by position variables $q^i(t)$ and action functional $S[q]$ [1] [2]

$$S[q] = \int_{t_1}^{t_2} dt\, L(q^i(t), \dot{q}^i(t), t) \tag{1.1}$$

A typical Lagrangian function is

$$L(\vec{q}, \dot{\vec{q}}) = \tfrac{1}{2}m\dot{\vec{q}}^2 - V(\vec{q}). \tag{1.2}$$

with mass m and $V(q)$ external potential.

A classical path extremises (minimises) the action S. Determine saddle-point $\delta S = 0$ by variation of the action [3]

$$
\begin{aligned}
\delta S &= \int_{t_1}^{t_2} dt \left(\delta q^i(t) \frac{\partial L}{\partial q^i} + \delta \dot{q}^i(t) \frac{\partial L}{\partial \dot{q}^i} \right) \\
&= \int_{t_1}^{t_2} dt\, \delta q^i(t) \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} \right) + \int_{t=t_1}^{t=t_2} d\left(\delta q^i(t) \frac{\partial L}{\partial \dot{q}^i} \right) \overset{!}{=} 0
\end{aligned}
\tag{1.3}
$$

First term is equation of motion (Euler–Lagrange)

$$\frac{\delta S}{\delta q^i(t)} = \frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = 0. \tag{1.4}$$

Second term due to partial integration is boundary e.o.m., usually ignore. [4]

Example.   Harmonic oscillator (free particle for $\omega = 0$)

$$L(\vec{q}, \dot{\vec{q}}) = \tfrac{1}{2}m\dot{\vec{q}}^2 - \tfrac{1}{2}m\omega^2\vec{q}^2, \qquad -m(\ddot{\vec{q}} + \omega^2\vec{q}) = 0. \tag{1.5}$$

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[1] L is often time-independent: $L(q^i, \dot{q}^i, t) = L(q^i, \dot{q}^i)$.

[2] A single time derivative $\dot{q}^i$ usually suffices.

[3] Einstein summation convention: there is an implicit sum over all values for pairs of equal upper/lower indices.

[4] Usually fix position $q_i(t_k) = $ const. (Dirichlet) or momentum $\partial L/\partial \dot{q}^i(t_k) = 0$ (Neumann) at boundary.

1.2 Hamiltonian Formulation

The Hamiltonian framework is the next step towards canonical quantum mechanics.

Define conjugate momentum p_i as[5]

$$p_i = \frac{\partial L}{\partial \dot{q}^i} \tag{1.6}$$

and solve for $\dot{q}^i = \dot{q}^i(q, p, t)$.[6] Define phase space as (q^i, p_i).

Lagrangian function $L(q, \dot{q}, t)$ replaced by Hamiltonian function $H(q, p, t)$ on phase space. Define $H(q^i, p_i, t)$ as Legendre transformation of L

$$H(q, p, t) = p_i \dot{q}^i(q, p, t) - L(q, \dot{q}(q, p, t), t). \tag{1.7}$$

Let us express e.o.m. through H. General variation reads

$$\delta H = \delta p_i \dot{q}^i - \delta q^i \frac{\partial L}{\partial q^i} \tag{1.8}$$

where we substituted definition of momenta p_i twice. Use Euler–Lagrange equation and momenta to simplify further

$$\delta H = \delta p_i \dot{q}^i - \delta q^i \dot{p}_i. \tag{1.9}$$

Now, Hamiltonian e.o.m. $\dot{q}^i = \partial H/\partial p_i$ and $\dot{p}_i = -\partial H/\partial q^i$.

Introduce Poisson brackets for functions f, g on phase space

$$\{f, g\} := \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i}. \tag{1.10}$$

Express time evolution for phase space functions $f(p, q, t)$ [7]

$$\frac{df}{dt} = \frac{\partial f}{\partial t} - \{H, f\}. \tag{1.11}$$

Works well for $f = q^i$ and $f = p_i$.

Example. Harmonic oscillator

$$\vec{p} = m\dot{\vec{q}}, \qquad H = \vec{p} \cdot \dot{\vec{q}} - \frac{m}{2}\dot{\vec{q}}^2 + \frac{m\omega^2}{2}\vec{q}^2 = \frac{1}{2m}\vec{p}^2 + \frac{m}{2}\omega^2\vec{q}^2. \tag{1.12}$$

Hamiltonian equations of motion

$$\dot{\vec{q}} = -\{H, \vec{q}\} = \frac{\partial H}{\partial \vec{p}} = \frac{1}{m}\vec{p}, \quad \dot{\vec{p}} = -\{H, \vec{p}\} = -\frac{\partial H}{\partial \vec{q}} = -m\omega^2\vec{q}. \tag{1.13}$$

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[5]This is a choice, could also use different factors or notations.

[6]Suppose the equation can be solved for $\dot{q}$.

[7]The Hamiltonian H is a phase space function.

Convenient change of variables

$$\vec{a} = \frac{1}{\sqrt{2m\omega}} \left(m\omega\vec{q} + i\vec{p} \right), \qquad \vec{a}^* = \frac{1}{\sqrt{2m\omega}} \left(m\omega\vec{q} - i\vec{p} \right), \tag{1.14}$$

with new Poisson brackets

$$\{f, g\} = -i \, \frac{\partial f}{\partial a^i} \frac{\partial g}{\partial a_i^*} + i \, \frac{\partial f}{\partial a_i^*} \frac{\partial g}{\partial a^i} \,. \tag{1.15}$$

Separated first-order time evolution for $\vec{a}, \vec{a}^\dagger$

$$H = \omega \vec{a}^\dagger \vec{a}, \qquad \dot{\vec{a}} = -i\omega\vec{a}, \quad \dot{\vec{a}}^\dagger = +i\omega\vec{a}^\dagger. \tag{1.16}$$

1.3 Quantum Mechanics

In canonical quantisation classical objects are replaced by elements of linear algebra:

- The state (q^i, p_i) becomes a vector $|\psi\rangle$ in a Hilbert space V.
- A phase space function f becomes a linear operator F on V.
- Poisson brackets $\{f, g\}$ become commutators $-i\hbar^{-1}[F, G]$.[8]

State equation of motion (Schrödinger), wave equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle. \tag{1.17}$$

Probabilistic role of wave function: $|\langle\phi|\psi\rangle|^2$ is probability. Requires:

- $\langle\psi|\psi\rangle$ is positive.
- $\langle\psi|\psi\rangle$ can be normalised to 1 by scaling $|\psi\rangle$.
- $\langle\psi|\psi\rangle$ is conserved

$$\frac{d}{dt}\langle\psi|\psi\rangle = (i\hbar)^{-1}\langle\psi|(H - H^\dagger)|\psi\rangle = 0. \tag{1.18}$$

  Hamiltonian is hermitian (self-adjoint). Unitary time evolution: $|\psi(t_2)\rangle = U(t_2, t_1)|\psi(t_1)\rangle$.
- $\langle\psi|F|\psi\rangle$ is expectation value of operator F. Obeys classical time evolution.

Example. Harmonic oscillator, free particle.

Momentum operator and Hamiltonian[9]

$$\vec{p} = -i\hbar\frac{\partial}{\partial\vec{q}}, \quad [q^i, p_j] = i\hbar\delta_j^i, \quad H = -\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial\vec{q}}\right)^2 + \frac{m\omega^2}{2}\vec{q}^2. \tag{1.19}$$

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[8] Cannot always be translated literally, but up to simpler terms.

[9] Note: $\vec{p} = -i\hbar\vec{\partial}\vec{q}$ vs. $E = +i\hbar\partial_t$. On wave function $|\psi\rangle = \int d^d\vec{q}\,\psi(\vec{q}, t)|\vec{q}\rangle$, however: $\vec{p}|\psi\rangle = +i\hbar\int d^d\vec{q}\,\vec{\partial}\psi(\vec{q}, t)|\vec{q}\rangle$ and $E|\psi\rangle = +i\hbar\int d^d\vec{q}\,\partial_t\psi(\vec{q}, t)|\vec{q}\rangle$

Free particle: momentum eigenstate (Fourier transforms)

$$|\vec{p}\rangle = \int d^d\vec{q}\, e^{-i\hbar^{-1}\vec{p}\cdot\vec{q}}|\vec{q}\rangle, \qquad |\vec{q}\rangle = \int \frac{d^d\vec{p}}{(2\pi\hbar)^d}\, e^{i\hbar^{-1}\vec{p}\cdot\vec{q}}|\vec{p}\rangle. \qquad (1.20)$$

$|\vec{p}\rangle$ is energy eigenstate with $E = \vec{p}^2/2m$.

Harmonic oscillator: use operators a^i and $a_i^\dagger$

$$\vec{a} = \frac{1}{\sqrt{2m\omega}}\left(m\omega\vec{q} + \hbar\frac{\partial}{\partial\vec{q}}\right), \quad \vec{a}^\dagger = \frac{1}{\sqrt{2m\omega}}\left(m\omega\vec{q} - \hbar\frac{\partial}{\partial\vec{q}}\right), \qquad (1.21)$$

with commutators

$$[a^i, a_j^\dagger] = \hbar\delta_j^i. \qquad (1.22)$$

Quantum Hamiltonian has extra vacuum energy $E_0 = \frac{1}{2}d\hbar\omega$

$$H = \tfrac{1}{2}\omega a^i a_i^\dagger + \tfrac{1}{2}\omega a_i^\dagger a^i = \omega\vec{a}^\dagger\vec{a} + \tfrac{1}{2}d\hbar\omega = \omega\vec{a}^\dagger\vec{a} + E_0. \qquad (1.23)$$

- Can add any E_0 to Hamiltonian. No effect. E_0 is irrelevant. Unless: E couples to something else (e.g. gravity).
- Same effect as adding $i\alpha(\vec{q}\cdot\vec{p} - \vec{p}\cdot\vec{q})$ to H.[10] Classically invisible. Quantum energy shift $\Delta E_0 = -d\alpha\hbar$. Quantum ordering ambiguity. Harmless, affects trivial E_0.
- Quantum theory does as it pleases, e.g. introduce/shift E_0. Best to consider all allowable terms in the first place.

Construct spectrum: Start from vacuum state $|0\rangle$ to be annihilated by $\vec{a}$ (has energy $E = E_0$, but irrelevant)

$$a^i|0\rangle = 0. \qquad (1.24)$$

Add excitations $n_i \geq 0$ and normalise state $\langle\vec{n}|\vec{n}\rangle = 1$

$$|\vec{n}\rangle = \left(\prod_{i=1}^d \frac{(a_i^\dagger)^{n_i}}{\sqrt{n_i}}\right)|0\rangle. \qquad (1.25)$$

Energy eigenstate with $E = \hbar\omega N + E_0$ where $N = \sum_{i=1}^d n_i$ is total excitation number. Crucial property

$$[H, \vec{a}^\dagger] = \omega\vec{a}^\dagger. \qquad (1.26)$$

1.4 Quantum Mechanics and Relativity

Let us set $\hbar = 1$, $c = 1$ for convenience.[11]

Attempts to set up a relativistic version of quantum mechanics have failed. Let us see why.

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[10]No ambiguity for $\vec{p}^2$ and $\vec{q}^2$, but useful to consider all second degree polynomials in $\vec{p}$ and $\vec{q}$.
[11]Can be recovered from considerations of physical units.

Non-relativistic and relativistic energy relation

$$e = \frac{\vec{p}^2}{2m}, \qquad \text{vs.} \qquad e^2 = \vec{p}^2 + m^2 \quad \text{or} \quad e = \sqrt{\vec{p}^2 + m^2}. \tag{1.27}$$

Natural guess for relativistic wave equation (Klein–Gordon)

$$\left(-\left(\frac{\partial}{\partial t}\right)^2 + \left(\frac{\partial}{\partial \vec{q}}\right)^2 - m^2 \right) |\psi\rangle = 0. \tag{1.28}$$

Has several conceptual problems:

Probabilistic Properties. The norm $\langle\psi|\psi\rangle$ of non-relativistic QM is conserved only for first-order wave equation.

There is a real conserved quantity

$$Q = \frac{i}{2m} \left(\langle\psi|\frac{\partial}{\partial t}|\psi\rangle - \frac{\partial}{\partial t}\langle\psi|\psi\rangle \right), \tag{1.29}$$

Problem:

- Q is not positive definite.
- Not suitable for probabilistic interpretation![12]

One can define a positive definite measure, but it is not local.

Why consider probabilities in a time slice in the first place?

Causality. Consider the overlap

$$\langle \vec{q}_2 | U(t_2, t_1) | \vec{q}_1 \rangle \tag{1.30}$$

for a pair of spacetime points (t_1, q_1) and (t_2, q_2). Probability amplitude for particle moving from 1 to 2.

Problem:

- Overlap non-zero if points are space-like separated.
- forbidden region: violation of causality?
- at least: exponential suppression (tunnelling).

Negative-Energy Solutions. Second-order wave equation. For every positive-energy solution

$$|\vec{p}, +, t\rangle = \int d^d\vec{q}\, e^{-i\vec{p}\cdot\vec{q} - ie(\vec{p})t} |\vec{q}\rangle \tag{1.31}$$

there is a negative-energy solution

$$|\vec{p}, -, t\rangle = \int d^d\vec{q}\, e^{-i\vec{p}\cdot\vec{q} + ie(\vec{p})t} |\vec{q}\rangle. \tag{1.32}$$

Problems:

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[12] As we shall see, Q is rather similar to an electric charge.

- Negative-energy particles not observed.[13]
- Positive-energy particle could fall to negative-energy state. A lot of energy released to produce other particles.

Could insist on positive energies by wave equation

$$i\,\frac{\partial}{\partial t}|\psi\rangle = \sqrt{-\left(\frac{\partial}{\partial \vec{q}}\right)^2 + m^2}\,|\psi\rangle. \qquad (1.33)$$

Problems:

- Square root of operator hard to define.
- Certainly non-local wave-equation.

Particle Creation.   Special relativity allows energy to be converted to rest mass of particles.

- Relativistic quantum mechanics should allow such processes.
- Quantum mechanics usually assumes a fixed particle number.

Dirac Equation.   The Dirac equation was an attempt to overcome some problems

$$\frac{\partial}{\partial t}|\psi\rangle = \alpha^k \frac{\partial}{\partial q^k}|\psi\rangle + \beta m|\psi\rangle. \qquad (1.34)$$

Relativistic wave equation; implies Klein–Gordon equation.

Probabilistic interpretation:

- First-order wave equation.
- $\langle\psi|\psi\rangle$ is conserved and positive definite.
- Positivity requires Bose statistics.

Spin:

- Operators α^k imply spin-1/2 particles.
- No spin-0 particles reproducible.
- Half-integer spin requires Fermi statistics.

Negative-energy solutions:

- Exist (with different spin d.o.f.).
- Separation from positive energies is non-local.

Dirac equation has the same problems as Klein–Gordon.

Conclusion.   Klein–Gordon and Dirac equations:

- Perfectly acceptable relativistic wave equations.
- No probabilistic interpretation.
- Model without particle production.

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[13]Extract energy from making particle faster!

1.5 Conventions

Units. We shall work with natural units $\hbar = c = 1$.

- $c = 299\ 792\ 458$ m s^{-1} therefore s := $299\ 792\ 458$ m.
- $\hbar = 1.055\ldots \times 10^{-34}$ kg m^2 s^{-1} therefore kg := $2.843\ldots \times 10^{42}$ m^{-1}.
- can always reinstall appropriate units by inserting $1 = c = \hbar$.
- particle physics unit electron Volt (eV): m = 5.068×10^6 eV^{-1},
  s = 1.519×10^{15} eV^{-1}, kg = 5.610×10^{35} eV.
- convert back to SI units:
  eV = 5.068×10^6 m^{-1} = 1.519×10^{15} s^{-1} = 1.783×10^{-36} kg.

Euclidean space. Write a three-vector x as

- x_j with Latin indices $k, l, \cdots = 1, 2, 3$.
- $\vec{x} = (x^1, x^2, x^3) = (x, y, z)$.

Scalar product between two vectors

$$\vec{a} \cdot \vec{b} := \sum_{k=1}^{3} a^k b^k = a^1 b^1 + a^2 b^2 + a^3 b^3. \tag{1.35}$$

Vector square

$$\vec{a}^{\,2} := \vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + a_3^2. \tag{1.36}$$

Totally anti-symmetric epsilon-tensor ε^{ijk} with normalisation

$$\varepsilon^{123} = +1. \tag{1.37}$$

Use to define cross product

$$(a \times b)^k = \varepsilon^{ijk} a^i b^j. \tag{1.38}$$

Minkowski Space. Four vectors, Greek indices $\mu, \nu, \ldots = 0, 1, 2, 3$:

- position vector $x^\mu := (x^0, x^1, x^2, x^3) = (t, \vec{x})$.
- momentum covector $p_\mu := (p_0, p_1, p_2, p_3) = (e, \vec{p})$.

Summation convention: repeated index μ means implicit sum over $\mu = 0, 1, 2, 3$

$$x^\mu p_\mu := \sum_{\mu=0}^{3} x^\mu p_\mu = et + \vec{x} \cdot \vec{p}. \tag{1.39}$$

Minkowski metric: signature $(-+++)$

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \text{diag}(-1, +1, +1, +1). \tag{1.40}$$

Raise and lower indices (wherever needed):

$$x_\mu := \eta_{\mu\nu} x^\mu = (-t, \vec{x}), \qquad p^\mu := \eta^{\mu\nu} p_\mu = (-e, \vec{p}). \tag{1.41}$$

Scalar products of two vectors or two covectors, e.g.

$$p \cdot p := -e^2 + \vec{p}^2. \tag{1.42}$$

Our conventions:

- Mass shell $p^2 = -m^2$: $p^2 < 0$ massive, $p^2 = 0$ massless, $p^2 > 0$ tachyonic.
- Light cone: $(x-y)^2 < 0$ time-like, $(x-y)^2 = 0$ light-like, $(x-y)^2 > 0$ space-like.

$$(1.43)$$
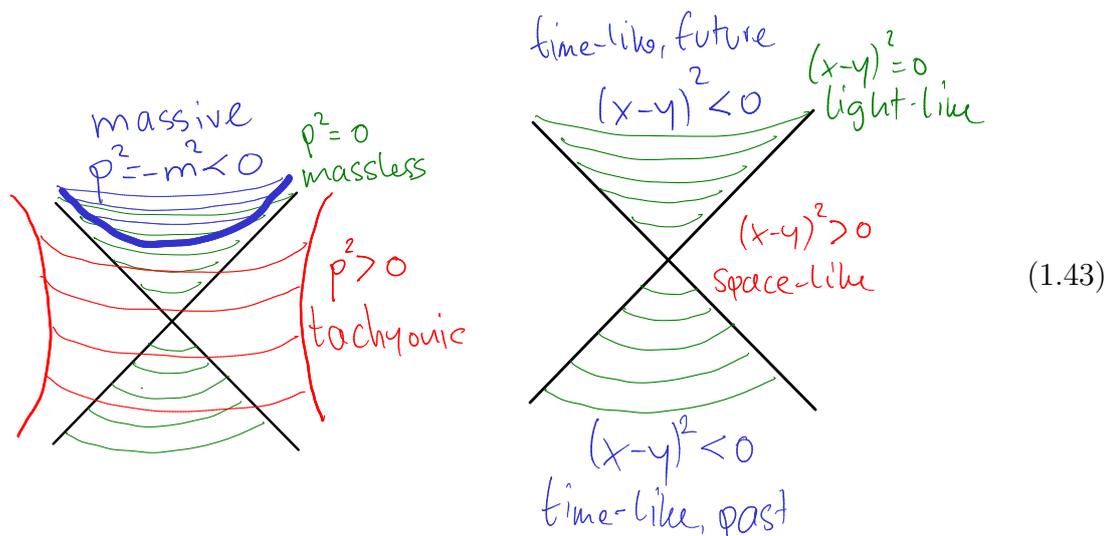
Why?

- notation follows space (not time). $\underline{x}^\mu = (t, \vec{\underline{x}})$ $\underline{p}_\mu = (t, \vec{\underline{x}})$
- $x^i = x_i$ but $x^0 = -x_0 = t$.
- $p^i = p_i$ but $p^0 = -p_0 = e$.
- Wick rotations natural: just rotate time $t \to it$ and obtain Euclidean metric.

How to convert?

- flip sign of every $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$.
- find out which (co)vectors match: x^μ and p_μ agree literally, x_μ and p^μ flip the sign.
- flip sign for every scalar product of vectors of same type: e.g. $p^2 + m^2 \leftrightarrow -p^2 + m^2$.
- preserve scalar product between different vectors: $x^\mu p_\mu$.[14]
- note: $\vec{p}$ opposite sign compared to Peskin & Schroeder; mild problem: sign of $\vec{p}$ and e is merely convention.

Name Spaces. We have only 26 Latin letters at our disposal and some are more attractive than others. Have to recycle:

- e may be $2.71\ldots$, but also energy,
- π may be $3.14\ldots$, but also momentum conjugate to field ϕ,
- i may be $\sqrt{-1}$, but also useful for counting.
- κ may look like k or K on the blackboard.

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[14]Therefore also $x \cdot p$ unchanged (two signs cancel).

- H may be Hamilton function or operator.
- ...

Will typically not say explicitly which letter means what:

- May even use same letter for different meanings in one formula.
- Can guess meaning from the context, e.g. i in $\exp(\pi i \dots)$ vs. $\sum_{i=1}^{n}$.
- Indices typically do not mix with other symbols.
- Could try to avoid, but may also clutter notation.
- It's a fact of life (and the literature).

2 Classical Free Scalar Field

In the following we shall discuss one of the simplest field theory models, the classical non-interacting relativistic scalar field.

2.1 Spring Lattice

Before considering field, start with an approximation we can certainly handle: lattice.

Consider an atomic lattice:

- 1D or 2D cubic lattice,
- atoms are coupled to neighbours by springs,[1]
- atoms are coupled to rest position by springs,
- atoms can move only orthogonally to lattice (transverse),
- boundaries: periodic identification.

$$\text{(2.1)}$$
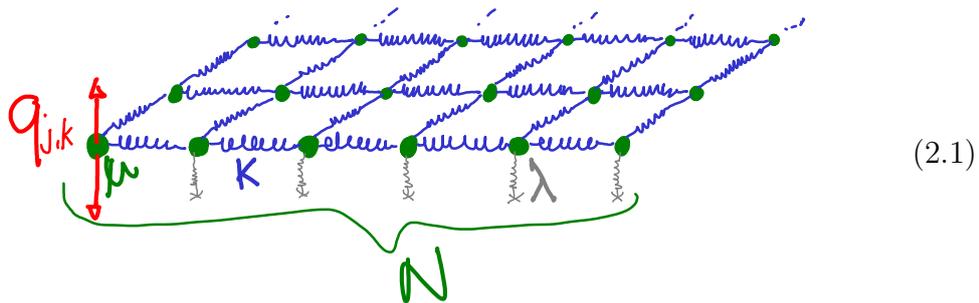
model parameters and variables:

- lattice separation r,
- number of atoms N (in each direction),
- mass of each atom μ,
- lattice spring constant κ,
- return spring constant λ,
- shift orthogonal to lattice $q_{j,k}$

Lagrangian Formulation.   Lagrange function

$$L = L_{\text{kin}} - V_{\text{lat}} - V_{\text{rest}} \tag{2.2}$$

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[1]Springs are useful approximations because they model first deviation from rest position; always applies to small excitations.

Standard non-relativistic kinetic terms

$$L_{\text{kin}} = \tfrac{1}{2}\mu \sum_{i,j=1}^{N} \dot{q}_{i,j}^2. \tag{2.3}$$

Potential for springs between atoms (ignore x, y-potential)[2]

$$V_{\text{lat}} = \tfrac{1}{2}\kappa \sum_{i,j=1}^{N} (q_{i-1,j} - q_{i,j})^2 + \tfrac{1}{2}\kappa \sum_{i,j=1}^{N} (q_{i,j-1} - q_{i,j})^2. \tag{2.4}$$

Some spring potential to drive atoms back to rest position

$$V_{\text{rest}} = \tfrac{1}{2}\lambda \sum_{i,j=1}^{N} q_{i,j}^2. \tag{2.5}$$

Quadratic in q's: bunch of coupled HO's. Equations of motion

$$\mu \ddot{q}_{i,j} \begin{array}{c} -\kappa(q_{i-1,j} - 2q_{i,j} + q_{i+1,j}) \\ -\kappa(q_{i,j-1} - 2q_{i,j} + q_{i,j+1}) \end{array} + \lambda q_{i,j} = 0. \tag{2.6}$$

Note: spatially homogeneous equations. Use discrete Fourier transform to solve (respect periodicity)

$$q_{i,j}(t) = \frac{1}{N^2} \sum_{k,l=1}^{N} \frac{c_{k,l}}{\sqrt{2\mu\omega_{k,l}}} \exp\left(-\frac{2\pi i}{N}(ki + lj) - i\omega_{k,l}t\right)$$

$$+ \frac{1}{N^2} \sum_{k,l=1}^{N} \frac{c_{k,l}^*}{\sqrt{2\mu\omega_{k,l}}} \exp\left(\frac{2\pi i}{N}(ki + lj) + i\omega_{k,l}t\right) \tag{2.7}$$

Used freedom to define coefficients $c_{k,l}$ to introduce prefactors. Complex conjugate coefficients $c_{k,l}^*$ in second term ensure reality of $q_{i,j}$. Note: $c_{k,l}^*$ represents mode of opposite momentum and energy w.r.t. $c_{k,l}$!

E.o.m. translates to dispersion relation:

$$\mu\omega_{k,l}^2 = \lambda + 4\kappa \sin^2 \frac{\pi k}{N} + 4\kappa \sin^2 \frac{\pi l}{N}. \tag{2.8}$$

$$\tag{2.9}$$

---
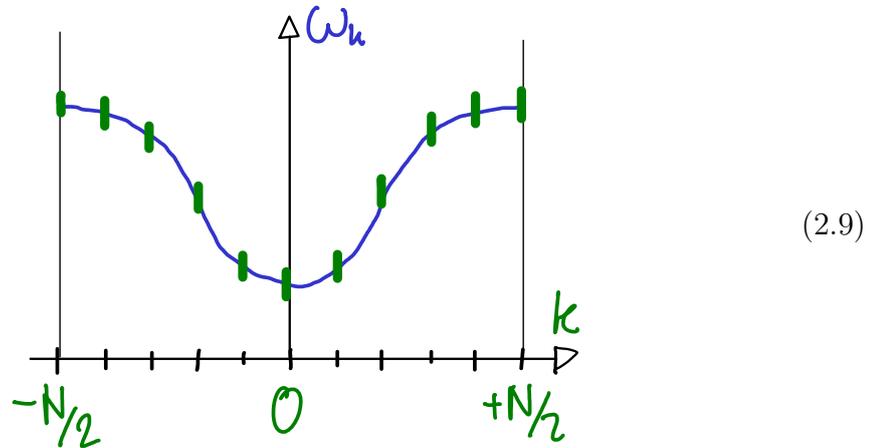
[2]Can ignore due to $d^2 = d_x^2 + d_y^2 + d_z^2$, shifts potential by an irrelevant constant. Moreover longitudinal and transverse excitations decouple.

Hamiltonian Formulation. Define momenta

$$p_{i,j} := \frac{\partial L}{\partial \dot{q}_{i,j}} = \mu \dot{q}_{i,j}. \tag{2.10}$$

Derive Hamiltonian function as Legendre transform of L

$$H = H_{\text{kin}} + V_{\text{lat}} + V_{\text{rest}} \quad \text{with} \quad H_{\text{kin}} = \frac{1}{2\mu} \sum_{i,j=1}^{N} p_{i,j}^2. \tag{2.11}$$

Then define canonical Poisson brackets

$$\{f, g\} := \sum_{i,j=1}^{N} \left(\frac{\partial f}{\partial q_{i,j}} \frac{\partial g}{\partial p_{i,j}} - \frac{\partial f}{\partial p_{i,j}} \frac{\partial g}{\partial q_{i,j}} \right). \tag{2.12}$$

In other words $\{q_{i,j}, p_{j,k}\} = \delta_{i,k}\delta_{j,l}$ and $\{q, q\} = \{p, p\} = 0$.

Fourier Modes. Introduce new complex variables (Fourier transform)

$$a_{k,l} = \frac{1}{\sqrt{2\mu\omega_{k,l}}} \sum_{i,j=1}^{N} \exp\left(\frac{2\pi i}{N}(ki + lj) \right) \left(\mu\omega_{k,l}q_{i,j} + ip_{i,j} \right). \tag{2.13}$$

Transformed Hamiltonian is very simple

$$H = \frac{1}{N^2} \sum_{k,l=1}^{N} \omega_{k,l} a_{k,l}^* a_{k,l}. \tag{2.14}$$

Poisson brackets for new variables are simple, too

$$\{a_{i,j}, a_{k,l}^*\} = -i\, N^2\, \delta_{i,k}\delta_{j,l},$$
$$\{a_{i,j}, a_{k,l}\} = \{a_{i,j}^*, a_{k,l}^*\} = 0. \tag{2.15}$$

Can convince oneself that equations of motion hold

$$\dot{a}_{k,l} = -\{H, a_{k,l}\} = -i\omega_{k,l}a_{k,l},$$
$$\dot{a}_{k,l}^* = -\{H, a_{k,l}^*\} = +i\omega_{k,l}a_{k,l}^*. \tag{2.16}$$

and solved by above Lagrangian solution

$$a_{k,l}(t) = c_{k,l} \exp(-i\omega_{k,l}t), \qquad a_{k,l}^*(t) = c_{k,l}^* \exp(+i\omega_{k,l}t). \tag{2.17}$$

2.2 Continuum Limit

Now turn this spring lattice into a smooth field φ:

- send number of sites $N \to \infty$.
- box of size L in all directions; lattice separation $r = L/N \to 0$.
- positions $x = ir = iL/N$,
- field $q_{i,\ldots} = \varphi(\vec{x})$,
- generalise to d spatial dimensions, e.g. $d = 1, 2, 3$.

Some useful rules

$$\sum_{i=1}^{N} \to \frac{1}{r} \int dx, \qquad q_i - q_{i-1} \to r(\partial\varphi) \tag{2.18}$$

Lagrangian Formulation. Substitute this in Lagrangian

$$L \to \int d^d \vec{x} \left(\frac{\mu}{2r^d} \dot{\varphi}^2 - \frac{\kappa}{2r^{d-2}} (\vec{\partial}\varphi)^2 - \frac{\lambda}{2r^d} \varphi^2 \right). \qquad (2.19)$$

Diverges as $r \to 0$, but can rescale parameters. Suitable rescalings

$$\mu = r^d \bar{\mu}, \quad \kappa = r^{d-2} \bar{\kappa}, \quad \lambda = r^d \bar{\lambda}, \qquad (2.20)$$

Parameters become densities. Lagrangian functional

$$L[\varphi, \dot{\varphi}](t) = \int d^d \vec{x} \left(\tfrac{1}{2} \bar{\mu} \dot{\varphi}^2 - \tfrac{1}{2} \bar{\kappa} (\vec{\partial}\varphi)^2 - \tfrac{1}{2} \bar{\lambda} \varphi^2 \right). \qquad (2.21)$$

Can furthermore rescale field $\varphi = \bar{\kappa}^{-1/2} \phi$

$$L[\phi, \dot{\phi}](t) = \int d^d \vec{x} \left(\tfrac{1}{2} \bar{\mu} \bar{\kappa}^{-1} \dot{\phi}^2 - \tfrac{1}{2} (\vec{\partial}\phi)^2 - \tfrac{1}{2} \bar{\lambda} \bar{\kappa}^{-1} \phi^2 \right). \qquad (2.22)$$

Derive e.o.m.: start with action functional $S[\phi]$

$$S[\phi] = \int dt \, L[\phi](t) = \int dt \, d^d \vec{x} \, \mathcal{L}\big(\phi(\vec{x}, t), \partial_i \phi(\vec{x}, t), \dot{\phi}(\vec{x}, t)\big); \qquad (2.23)$$

useful to express (homogeneous) Lagrange functional $L[\phi](t)$ through Lagrangian density $\mathcal{L}(\phi, \vec{\partial}\phi, \dot{\phi})$ (the Lagrangian).

Vary action functional (discard boundary terms)

$$\delta S[\phi] = \int dt \, d^d \vec{x} \, \delta\phi \left(\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^i} \frac{\partial \mathcal{L}}{\partial(\partial_i \phi)} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) + \dots \overset{!}{=} 0. \qquad (2.24)$$

Write general Euler–Lagrange equation for fields

$$\frac{\partial \mathcal{L}}{\partial \phi}(\vec{x}, t) - \frac{\partial}{\partial x^i} \frac{\partial \mathcal{L}}{\partial(\partial_i \phi)}(\vec{x}, t) - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}}(\vec{x}, t) = 0. \qquad (2.25)$$

In our case

$$-\bar{\mu} \bar{\kappa}^{-1} \ddot{\phi} + \vec{\partial}^2 \phi - \bar{\lambda} \bar{\kappa}^{-1} \phi = 0; \qquad (2.26)$$

agrees with continuum limit of discrete e.o.m.. Now denote as speed of light c and mass m

$$\bar{\mu} \bar{\kappa}^{-1} = c^{-2} = 1, \qquad \bar{\lambda} \bar{\kappa}^{-1} = m^2, \qquad (2.27)$$

to discover Klein–Gordon equation (set $c = 1$)

$$-c^{-2} \ddot{\phi} + \vec{\partial}^2 \phi - m^2 \phi = 0. \qquad (2.28)$$

Plane Wave Solutions. Consider solutions on infinite space and time. Homogeneous equation solved by Fourier transformation

$$\phi(\vec{x}, t) = \int \frac{d^d\vec{p}}{(2\pi)^d \, 2e(\vec{p})} \, \alpha(\vec{p}) \exp\!\big(-i\vec{p}\!\cdot\!\vec{x} - ie(\vec{p})t\big)$$
$$+ \int \frac{d^d\vec{p}}{(2\pi)^d \, 2e(\vec{p})} \, \alpha^*(\vec{p}) \exp\!\big(+i\vec{p}\!\cdot\!\vec{x} + ie(\vec{p})t\big) \tag{2.29}$$

with (positive) energy $e(\vec{p})$ on mass shell (often called $\omega(\vec{p})$)

$$e(\vec{p}) = \sqrt{\vec{p}^2 + m^2}. \tag{2.30}$$

Agrees with discrete solution identifying momenta as

$$p = 2\pi k/L, \qquad \sum_{k=1}^{N} \to \frac{L}{2\pi} \int dp, \qquad c_{k,\ldots} = \frac{\alpha(\vec{p})}{\sqrt{2e(\vec{p})r^d}}. \tag{2.31}$$

Some remarks on factors and conventions:

- Fourier transforms on $\mathbb{R}$ produce factors of 2π, need to be put somewhere. Convention to associate $(2\pi)^{-1}$ to every dp: $đp := dp/2\pi$. No factors of 2π for dx. No factors of 2π in exponent.
- Combination $d^d\vec{p}/2e(\vec{p})$ is a useful combination: Relativistic covariance. Reason for conversion factor in $c_{k,\ldots}$.

2.3 Relativistic Covariance

The Klein–Gordon equation can be written manifestly relativistically[3] [4]

$$-\partial^\mu\partial_\mu\phi + m^2\phi = -\partial^2\phi + m^2\phi = 0. \tag{2.32}$$

Also Lagrangian and action manifestly relativistic

$$\mathcal{L} = -\tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}m^2\phi^2, \qquad S = \int d^D x \, \mathcal{L}. \tag{2.33}$$

To understand the relativistic behaviour of the solution, consider integration over a mass shell $p^2 + m^2 = 0$

$$\int d^D p \, \delta(p^2 + m^2)\theta(p^0) \, f(p)$$
$$= \int d^d\vec{p} \, de \, \delta(-e^2 + \vec{p}^2 + m^2)\theta(e) \, f(e, \vec{p})$$
$$= \int \frac{d^d\vec{p} \, de}{2e} \, \delta(e - \sqrt{\vec{p}^2 + m^2})\theta(e) \, f(e, \vec{p})$$
$$= \int \frac{d^d\vec{p}}{2e(\vec{p})} \, f(e(\vec{p}), \vec{p}) \tag{2.34}$$

---

[3] ∂^2 often written as D'Alembertian $\Box$.
[4] Signature of spacetime is $-+++$!

Solution is just Fourier transform

$$\phi(x) = \int \frac{d^D p}{(2\pi)^D} \, \phi(p) \exp\left(-i p_\mu x^\mu\right) \tag{2.35}$$

with momentum space field defined on shell only

$$\phi(p) = 2\pi \delta(p^2 + m^2)\left(\theta(p_0)\alpha(\vec{p}) + \theta(-p_0)\alpha^*(-\vec{p})\right). \tag{2.36}$$

Momentum space e.o.m. obviously satisfied

$$(p^2 + m^2)\phi(p) = 0. \tag{2.37}$$

$\alpha(\vec{p})$ and $\alpha^*(\vec{p})$ define amplitudes on forward/backward mass shells.
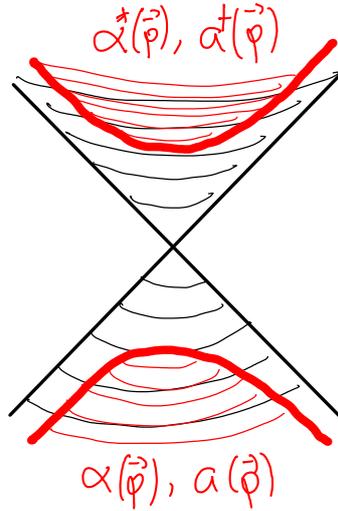
$$\tag{2.38}$$

Note that $\phi(p)$ obeys reality condition (from $\phi(x)^* = \phi(x)$)

$$\phi(p)^* = \phi(-p). \tag{2.39}$$

2.4 Hamiltonian Field Theory

Now that we have a nice relativistic formulation for the Klein–Gordon field $\phi(x)$, let's separate space from time. [5]

Position Space. Define momentum π (field) conjugate to ϕ

$$\pi(\vec{x}, t) = \frac{\delta L}{\delta \dot{\phi}(\vec{x})}(t) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}(\vec{x}, t) = \dot{\phi}(\vec{x}, t). \tag{2.40}$$

Determine Hamiltonian

$$\begin{aligned}
H[\phi, \pi] &= \int d^d \vec{x} \, \pi \dot{\varphi} - L[\phi, \dot{\phi}] \\
&= \int d^d \vec{x} \left(\tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\vec{\partial}\phi)^2 + \tfrac{1}{2}m^2\phi^2 \right).
\end{aligned} \tag{2.41}$$

---

[5] Formalism breaks relativistic invariance, physics remains relativistic.

Not relativistically covariant, not designed to be.[6]

Define Poisson brackets for phase space functionals f, g

$$\{f, g\} = \int d^d \vec{x} \left(\frac{\delta f}{\delta \phi(\vec{x})} \frac{\delta g}{\delta \pi(\vec{x})} - \frac{\delta f}{\delta \pi(\vec{x})} \frac{\delta g}{\delta \phi(\vec{x})} \right). \tag{2.42}$$

Poisson brackets of fields yield delta-functions[7]

$$\{\phi(\vec{x}), \pi(\vec{y})\} = \int d^d \vec{z}\, \delta^d(\vec{x} - \vec{z}) \delta^d(\vec{y} - \vec{z}) = \delta^d(\vec{x} - \vec{z}). \tag{2.43}$$

Momentum Space.   Now introduce momentum modes[8]

$$a(\vec{p}) = \int d^d \vec{x} \exp\left(i \vec{p} \cdot \vec{x}\right) \left(e(\vec{p}) \phi(\vec{x}) + i \pi(\vec{x})\right), \tag{2.44}$$

and inverse Fourier transformation

$$\phi(\vec{x}) = \int \frac{d^d \vec{p}}{(2\pi)^d\, 2e(\vec{p})}\, a(\vec{p}) \exp(-i \vec{p} \cdot \vec{x})$$

$$+ \int \frac{d^d \vec{p}}{(2\pi)^d\, 2e(\vec{p})}\, a^*(\vec{p}) \exp(+i \vec{p} \cdot \vec{x}),$$

$$\pi(\vec{x}) = -\frac{i}{2} \int \frac{d^d \vec{p}}{(2\pi)^d}\, a(\vec{p}) \exp(-i \vec{p} \cdot \vec{x})$$

$$+ \frac{i}{2} \int \frac{d^d \vec{p}}{(2\pi)^d}\, a^*(\vec{p}) \exp(+i \vec{p} \cdot \vec{x}). \tag{2.45}$$

Compute Poisson brackets for Fourier modes[9] [10]

$$\{a(\vec{p}), a^*(\vec{q})\} = -i\, 2e(\vec{p})\, (2\pi)^d \delta^d(\vec{p} - \vec{q}). \tag{2.46}$$

In other words

$$\{f, g\} = -i(2\pi)^d \int d^d \vec{p}\, 2e(\vec{p}) \left(\frac{\delta f}{\delta a(\vec{p})} \frac{\delta g}{\delta a^*(\vec{p})} - \frac{\delta f}{\delta a^*(\vec{p})} \frac{\delta g}{\delta a(\vec{p})} \right). \tag{2.47}$$

Hamiltonian translates to

$$H = \frac{1}{2} \int \frac{d^d \vec{p}}{(2\pi)^d}\, a^*(\vec{p}) a(\vec{p}) = \int \frac{d^d \vec{p}}{(2\pi)^d\, 2e(\vec{p})}\, e(\vec{p}) a^*(\vec{p}) a(\vec{p}). \tag{2.48}$$

Hence e.o.m. for field oscillator

$$\dot{a}(\vec{p}) = -\{H, a(\vec{p})\} = -ie(\vec{p}) a(\vec{q}),$$
$$\dot{a}^*(\vec{p}) = -\{H, a^*(\vec{p})\} = +ie(\vec{p}) a^*(\vec{q}). \tag{2.49}$$

One HO for every momentum. Solution

$$a(\vec{p}, t) = \alpha(\vec{p}) \exp(-ie(\vec{p})t), \quad a^*(\vec{p}, t) = \alpha^*(\vec{p}) \exp(+ie(\vec{p})t). \tag{2.50}$$

---

[6]Hamiltonian governs *time* translation.

[7]Formula for variation $\delta\phi(\vec{x})/\delta\phi(\vec{z}) = \delta^d(\vec{x} - \vec{z})$.

[8]Have some additional factors compared to some literature.

[9]Conventional 2π for delta-function in momentum space.

[10]Factor $2e(\vec{p})$ appropriate relativistic measure for mass shell.

3 Scalar Field Quantisation

We can now go ahead and try to quantise the classical scalar field using the canonical procedure described before. We will encounter some infinities, and discuss how to deal with them. Then we shall investigate a few basic objects in QFT.

3.1 Quantisation

Start with the Hamiltonian formulation of the scalar field discussed earlier.

Equal-Time Commutators. Phase space consists of the field $\phi(\vec{x})$ and the conjugate momentum $\pi(\vec{x})$ with Poisson bracket[1]

$$\{\phi(\vec{x}), \pi(\vec{y})\} = \delta^d(\vec{x} - \vec{y}). \tag{3.1}$$

Hence the canonical quantisation implies operators $\phi(\vec{x})$ and $\pi(\vec{x})$

$$[\phi(\vec{x}), \pi(\vec{y})] = i\delta^d(\vec{x} - \vec{y}). \tag{3.2}$$

Note that ϕ and π are now operator-valued fields (rather: distributions).[2]

Field States. Next have to define some states. Straight QM would lead to a state $|\phi_0\rangle$ for every field configuration $\phi_0(\vec{x})$ such that

$$\phi(\vec{x})|\phi_0\rangle = \phi_0(\vec{x})|\phi_0\rangle, \qquad \pi(\vec{x})|\phi_0\rangle = -i\frac{\delta}{\delta\phi_0(\vec{x})}|\phi_0\rangle. \tag{3.3}$$

Can be done formally, but not so convenient. Sketch of an eigenstate

$$|0\rangle = \int D\phi \, \exp\left(-\frac{1}{2}\int d^d\vec{x}\, d^d\vec{y}\, \Omega(\vec{x}, \vec{y})\, \phi(\vec{x})\, \phi(\vec{y})\right). \tag{3.4}$$

---

[1]At the moment, the fields are defined on a common time slice t, e.g. $t = 0$. Later we discuss unequal times.

[2]The delta-function is a distribution, also the fields should be considered distributions. Distributions are linear maps from test functions to numbers (or operators in this case). In physics, we write them as integrals with a distributional kernel (e.g. the delta-function). Sometimes we also perform illegal operations (e.g. evaluate delta-function $\delta(x)$ at $x = 0$).

Momentum Space. The classical field is a bunch of coupled HO's, let us diagonalise them and use creation and annihilation operators.

Go to momentum space, and pick a and $a^\dagger$ appropriately, same transformation as above

$$
a(\vec{p}) = \int d^d\vec{x} \exp(i\vec{p}\cdot\vec{x}) \left(e(\vec{p})\phi(\vec{x}) + i\pi(\vec{x}) \right),
$$

$$
\phi(\vec{x}) = \int \frac{d^d\vec{p}}{(2\pi)^d \, 2e(\vec{p})} \, a(\vec{p}) \exp(-i\vec{p}\cdot\vec{x})
$$

$$
+ \int \frac{d^d\vec{p}}{(2\pi)^d \, 2e(\vec{p})} \, a^\dagger(\vec{p}) \exp(+i\vec{p}\cdot\vec{x}),
$$

$$
\pi(\vec{x}) = -\frac{i}{2} \int \frac{d^d\vec{p}}{(2\pi)^d} \, a(\vec{p}) \exp(-i\vec{p}\cdot\vec{x})
$$

$$
+ \frac{i}{2} \int \frac{d^d\vec{p}}{(2\pi)^d} \, a^\dagger(\vec{p}) \exp(+i\vec{p}\cdot\vec{x}). \tag{3.5}
$$

Commutation relations in momentum space

$$
[a(\vec{p}), a^\dagger(\vec{q})] = 2e(\vec{p}) \, (2\pi)^d \delta^d(\vec{p} - \vec{q}). \tag{3.6}
$$

Substitute fields into Hamiltonian paying attention to ordering

$$
H = \frac{1}{4} \int \frac{d^d\vec{p}}{(2\pi)^d} \left(a^\dagger(\vec{p})a(\vec{p}) + a(\vec{p})a^\dagger(\vec{p}) \right)
$$

$$
= \frac{1}{4} \int \frac{d^d\vec{p}}{(2\pi)^d} \left(2a^\dagger(\vec{p})a(\vec{p}) + [a(\vec{p}), a^\dagger(\vec{p})] \right)
$$

$$
= \frac{1}{2} \int \frac{d^d\vec{p}}{(2\pi)^d} \, a^\dagger(\vec{p})a(\vec{p}) + \frac{1}{2} \int d^d\vec{p} \, e(\vec{p}) \, \delta^d(\vec{p} - \vec{p}). \tag{3.7}
$$

Vacuum Energy. Introduce a vacuum state $|0\rangle$ annihilated by all $a(\vec{p})$. Two problems with vacuum energy $H|0\rangle = E_0|0\rangle$

$$
E_0 = \frac{1}{2} \int d^d\vec{p} \, e(\vec{p}) \, \delta^d(\vec{p} - \vec{p}) : \tag{3.8}
$$

- delta-function is $\delta^d(\vec{p} - \vec{p}) = \delta^d(0)$ is ill-defined,
- integral $\frac{1}{2} \int d^d\vec{p} \, e(\vec{p})$ diverges.

These are self-made problems:

- We considered an infinite volume. Not reasonable to expect a finite energy. IR problem! The delta-function has units of volume. It measures the volume of the system $\delta^d(\vec{p} - \vec{p}) \sim V \to \infty$. Consider energy density instead!
- Integral $\frac{1}{2} \int_0^P d^d\vec{p} \, e(\vec{p})$ represents vacuum energy density. Infinitely many oscillators per volume element, not reasonable to expect finite energy density. UV problem! Introduce cutoff $|\vec{p}| < P$ to obtain $\frac{1}{2} \int_0^P d^d\vec{p} \, e(\vec{p}) \sim P^{d+1}$.

Finite lattice has similar effect to regularise IR and UV.

To avoid IR infinities in QFT:

- Put the system in a finite box. Possible but inconvenient. Rather work with infinite system and cope with IR divergences. E.g. consider energy density.

To avoid UV infinities in QFT:

- By definition we want a *field* theory, not a discrete model. Physics: Should be able to approximate by discrete model.
- Regularisation: impose momentum cut-off, consider a lattice, other tricks without physical motivation, ...
- Renormalisation: remove terms which would diverge.
- Remove regularisation: obtain finite results.

Here: simply drop E_0.

- There is no meaning to an absolute vacuum energy.
- May add any constant to Hamiltonian (to compensate E_0).
- Makes no observable difference in any physical process.
- Vacuum energy only a philosophical or religious question.
- Later on, infinities lead to interesting effects.

Renormalised Hamiltonian H_{ren}

$$H_{\mathrm{ren}} := H - E_0 = \frac{1}{2} \int \frac{d^d \vec{p}}{(2\pi)^d} \, a^\dagger(\vec{p}) a(\vec{p}). \qquad (3.9)$$

Nice, vacuum has zero energy.

Normal Ordering.   The ordering of variables in classical H plays no role. In quantum theory it does! It is responsible for vacuum energy E_0.

How to map some classical observable O to a quantum operator?

A possible map is normal ordering $\mathrm{N}(O)$:

- Express O in terms of a and $a^* \to a^\dagger$.
- Write all $a^\dagger$'s to the left of all a's.

Here, renormalised Hamiltonian is the normal ordering of H

$$H_{\mathrm{ren}} = \mathrm{N}(H). \qquad (3.10)$$

There are other ordering prescriptions for operators:

- Normal ordering depends on the choice of vacuum state. There may be other normal-ordering prescriptions associated to different "vacuum" states.
- There are other useful ordering prescriptions which we will encounter, e.g. time ordering.

3.2   Fock Space

We have:

- a collection of HO's labelled by momentum $\vec{p}$,

- a vacuum state $|0\rangle$ with energy $E = 0$.

Discuss other related states.

Single-Particle States. Can now excite vacuum

$$|\vec{p}\rangle := a^\dagger(\vec{p})|0\rangle. \tag{3.11}$$

Energy eigenstate with energy

$$E = +e(\vec{p}). \tag{3.12}$$

Energy of a particle with momentum $\vec{p}$ and mass m.

Negative-Energy Solutions. Note:
- Energy is positive $E > 0$.
- State $a(\vec{p})|0\rangle$ would have negative energy. Does not exist by construction.

Problem of negative-energy particles solved!
- $a^\dagger(\vec{p})$ creates a particle of momentum $\vec{p}$ a positive energy $+e(\vec{p})$ from the vacuum. $a^\dagger(\vec{p})$ is particle creation operator.
- $a(\vec{p})$ removes a particle of momentum $\vec{p}$ and thus removes the positive energy $+e(\vec{p})$ from the state.

$$a(\vec{p})|\vec{q}\rangle = a(\vec{p})a^\dagger(\vec{q})|0\rangle = [a(\vec{p}), a^\dagger(\vec{q})]|0\rangle$$
$$= 2e(\vec{p})\,(2\pi)^d\,\delta^d(\vec{p} - \vec{q})\,|0\rangle. \tag{3.13}$$

$a(\vec{p})$ is particle annihilation operator.[3]
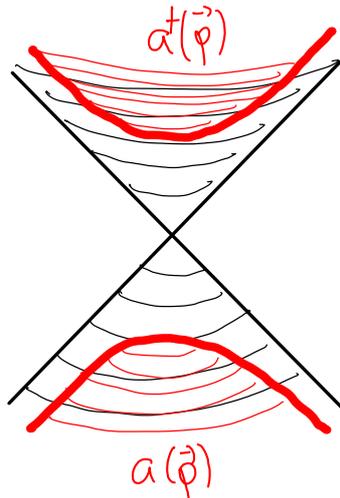
$$(3.14)$$

Interpretation of position space operators not as straight:
$\phi(\vec{x}) = \phi^\dagger(\vec{x})$ as well as $\pi(\vec{x}) = \pi^\dagger(\vec{x})$ create *or* annihilate a particle at position $\vec{x}$. Result is typically superposition.

---

[3] $a(\vec{p})$ is not an anti-particle (although this is sometimes claimed). It has negative energy while anti-particles (like particles) have positive energy. For the real scalar, the particle is its own anti-particle.

Normalisation.   Let us define proper normalisation for the vacuum

$$\langle 0|0\rangle = 1. \tag{3.15}$$

Normalisation of a single-particle state:

$$\langle \vec{p}|\vec{p}\rangle = 2e(\vec{p})\,(2\pi)^d\,\delta^d(\vec{p}-\vec{p}) = \infty. \tag{3.16}$$

Known problem from QM: Plane-wave states have infinite extent, smeared over all space, unphysical. Recall $\delta^d(\vec{p}-\vec{p})$ represents volume of space.

Consider peaked wave packet state $|f\rangle$ (test function) instead

$$|f\rangle := \int \frac{d^d p\, f(\vec{p})}{(2\pi)^d\, 2e(\vec{p})}\,|\vec{p}\rangle. \tag{3.17}$$

For suitable $f(\vec{p})$ this state has a finite normalisation

$$\langle f|f\rangle = \int \frac{d^d p\,|f(\vec{p})|^2}{(2\pi)^d\, 2e(\vec{p})}\,. \tag{3.18}$$

Multi-Particle States and Fock Space.   Now excite more than one harmonic oscillator

$$|\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n\rangle := a^\dagger(\vec{p}_1)\, a^\dagger(\vec{p}_2) \,\cdots\, a^\dagger(\vec{p}_n)\,|0\rangle. \tag{3.19}$$

The energy is given by the sum of particle energies[4]

$$E = \sum_{k=1}^{n} e(\vec{p}_k). \tag{3.20}$$

All particles are freely interchangeable

$$|\ldots, \vec{p}, \vec{q}, \ldots\rangle = |\ldots, \vec{q}, \vec{p}, \ldots\rangle \tag{3.21}$$

because creation operators commute

$$[a^\dagger(\vec{p}), a^\dagger(\vec{q})] = 0. \tag{3.22}$$

Bose statistics for indistinguishable particles: Wave function automatically totally symmetric in QFT.

A generic QFT state (based on the vacuum $|0\rangle$) is a linear combination of k-particle states with k not fixed. This vector space is called Fock space. It is the direct sum

$$\mathbb{V}_{\text{Fock}} = \mathbb{V}_0 \oplus \mathbb{V}_1 \oplus \mathbb{V}_2 \oplus \mathbb{V}_3 \oplus \ldots, \qquad \mathbb{V}_n = (\mathbb{V}_1)^{\otimes_{\text{s}} n} \tag{3.23}$$

of n-particle spaces $\mathbb{V}_n$ where

- $\mathbb{V}_0 = \mathbb{C}$ merely contains the vacuum state $|0\rangle$.

---

[4]Use $[H, a^\dagger(\vec{p})] = e(\vec{p})a^\dagger(\vec{p})$ to show $H|n\rangle = Ha_1^\dagger\ldots a_n^\dagger|0\rangle = [H, a_1^\dagger]a_2^\dagger\ldots a_n^\dagger|0\rangle + \ldots + a_1^\dagger\ldots a_{n-1}^\dagger[H, a_n^\dagger]|0\rangle = E|n\rangle.$

- $\mathbb{V}_1 = \mathbb{V}_{\text{particle}}$ is the space of single particle states $|\vec{p}\rangle$ with positive energy.
- $\mathbb{V}_n$ is the symmetric tensor product of n copies of $\mathbb{V}_1$.

Consider non-relativistic physics:

- The available energy is bounded from above. Much smaller than particle rest masses $m = e(0)$.
- Relevant part of Fock space with n bounded. For example: $\mathbb{V}_1$, $\mathbb{V}_2$ or $\mathbb{V}_1 \oplus \mathbb{V}_2$.
- Multiple-particle QM is a low-energy limit of QFT. Becomes QFT when number of particles is unbounded.

Conservation Laws. The Hamiltonian H measures the total energy E.

There is also a set of operators to measure total momentum $\vec{P}$

$$\vec{P} := \int \frac{d^d\vec{p}}{(2\pi)^d \, 2e(\vec{p})} \, \vec{p} \, a^\dagger(\vec{p}) a(\vec{p}). \tag{3.24}$$

with eigenvalue $\vec{P} = \sum_{k=1}^{n} \vec{p}_k$ on state $|\vec{p}_1, \ldots \vec{p}_n\rangle$. Vacuum carries no momentum. Combine into relativistic vector $P_\mu = (H, \vec{P})$ of operators

$$P_\mu := \int \frac{d^d\vec{p}}{(2\pi)^d \, 2e(\vec{p})} \, p_\mu(\vec{p}) \, a^\dagger(\vec{p}) a(\vec{p}), \qquad p_\mu(\vec{p}) := (e(\vec{p}), \vec{p}). \tag{3.25}$$

Another useful operator is the particle number operator n

$$N := \int \frac{d^d\vec{p}}{(2\pi)^d \, 2e(\vec{p})} \, a^\dagger(\vec{p}) a(\vec{p}). \tag{3.26}$$

It measures the number of particles n in a state $|\vec{p}_1, \ldots \vec{p}_n\rangle$

$$N\mathbb{V}_n = n\mathbb{V}_n. \tag{3.27}$$

The relativistic momentum vector and the number operator are conserved

$$[H, P_\mu] = [H, N] = 0. \tag{3.28}$$

Moreover, they carry no momentum

$$[P_\mu, P_\nu] = [P_\mu, N] = 0. \tag{3.29}$$

In fact, there are many conservation laws. Any two operators made from number density operators commute

$$n(\vec{p}) := a^\dagger(\vec{p}) a(\vec{p}), \qquad [n(\vec{p}), n(\vec{q})] = 0. \tag{3.30}$$

Hence such operators are conserved, carry no momentum and no particle number:

$$[H, n(\vec{p})] = [P_\mu, n(\vec{p})] = [N, n(\vec{p})] = 0. \tag{3.31}$$

In a free theory there are infinitely many conservation laws.[5]

---

[5]Should not say this: Free theory is trivial.

3.3 Complex Scalar Field

Let us discuss a slightly more elaborate case of the scalar field, the complex scalar, where we first encounter anti-particles.

The complex scalar field $\phi(x)$ has the Lagrangian[6] [7]

$$\mathcal{L} = -\partial^\mu \phi^* \, \partial_\mu \phi - m^2 \phi^* \phi = -|\partial \phi|^2 - m^2 |\phi|^2. \tag{3.32}$$

For the conjugate momentum we obtain[8]

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^*, \qquad \pi^* = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^*} = \dot{\phi}. \tag{3.33}$$

In the quantum theory we then impose the canonical commutators

$$[\phi(\vec{x}), \pi(\vec{y})] = [\phi(\vec{x})^\dagger, \pi(\vec{y})^\dagger] = i\delta(\vec{x} - \vec{y}). \tag{3.34}$$

The equation of motion associated to the above Lagrangian is the very same Klein–Gordon equation. However, now complex solutions $\phi(x)$ are allowed. Field operators (with time dependence, see below) now read:

$$\phi(x) = \int \frac{d^d \vec{p}}{(2\pi)^d \, 2e(\vec{p})} \, b(\vec{p}) \exp(-ip{\cdot}x)$$

$$+ \int \frac{d^d \vec{p}}{(2\pi)^d \, 2e(\vec{p})} \, a^\dagger(\vec{p}) \exp(+ip{\cdot}x),$$

$$\phi^\dagger(x) = \int \frac{d^d \vec{p}}{(2\pi)^d \, 2e(\vec{p})} \, a(\vec{p}) \exp(-ip{\cdot}x)$$

$$+ \int \frac{d^d \vec{p}}{(2\pi)^d \, 2e(\vec{p})} \, b^\dagger(\vec{p}) \exp(+ip{\cdot}x), \tag{3.35}$$

Note the strange appearance of a and b. For $a \neq b$ we have $\phi \neq \phi^\dagger$ while $a = b$ implies a real field $\phi = \phi^\dagger$. Non-trivial commutation relations:

$$[a(\vec{p}), a^\dagger(\vec{q})] = [b(\vec{p}), b^\dagger(\vec{q})] = 2e(\vec{p}) \, (2\pi)^d \delta^d(\vec{p} - \vec{q}). \tag{3.36}$$

and quantum Hamiltonian

$$H_{\text{ren}} := \frac{1}{2} \int \frac{d^d \vec{p}}{(2\pi)^d} \left(a^\dagger(\vec{p}) a(\vec{p}) + b^\dagger(\vec{p}) b(\vec{p}) \right). \tag{3.37}$$

The complex scalar carries a charge, $+1$ for ϕ and -1 for $\phi^\dagger$.

---

[6]The prefactor $\frac{1}{2}$ of the real scalar field (ϕ^2) is now absent. It is a convenient symmetry factor. Here the appropriate symmetry factor is 1 because $\phi^* \neq \phi$. More on symmetry factors later.

[7]One can also set $\phi(x) = (\phi_1 + i\phi_2)/\sqrt{2}$ and $\phi^*(x) = (\phi_1 - i\phi_2)/\sqrt{2}$ and obtain two independent scalar fields with *equal mass*.

[8]One may just as well define $\pi = \partial \mathcal{L}/\partial \dot{\phi}^* = \dot{\phi}$. It is a matter of convention, and makes no difference if applied consistently.

- The operator $a^\dagger$ creates a particle with charge $+1$.
- The operator b has negative energy, it should remove a particle. The operator carries the same charge as $a^\dagger$, hence the corresponding particle should have charge -1.

Two types of particles: particle and anti-particle. Opposite charges, but equal mass and positive energies. Conclusion:

- $a^\dagger$ creates a particle,
- b annihilates an anti-particle,
- $b^\dagger$ creates an anti-particle,
- a annihilates a particle,
- vacuum annihilated by a's and b's.
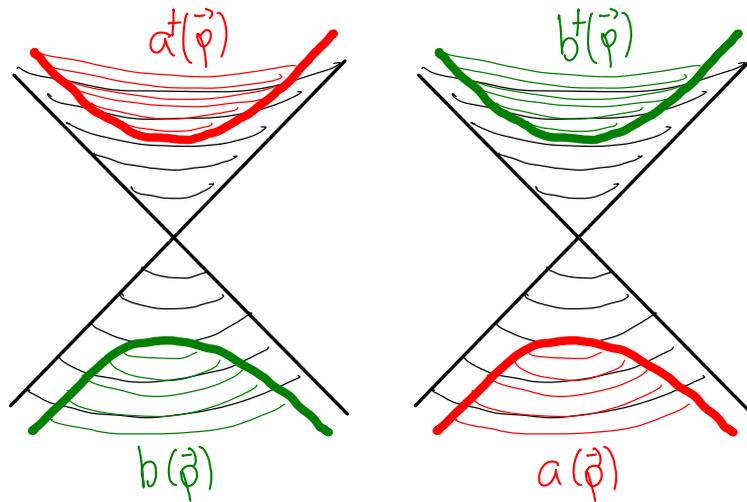
$$(3.38)$$

3.4 Correlators

Have quantised field. States have an adjustable number of indistinguishable particle with definite momentum. Now what? Consider particle propagation in space and time.

Schrödinger Picture. Steps:

- create a particle at $x^\mu = (t, \vec{x})$,
- let the state evolve for some time $s - t$,
- measure the particle at $y^\mu = (s, \vec{y})$.

Could use $\phi(\vec{x})$ or $\pi(\vec{x})$ to create particle from vacuum.[9] Use ϕ because $\pi = \dot\phi$ can be obtained from time derivative.

In Schrödinger picture the states evolve in time

$$i\frac{d}{dt}\ket{\Psi, t} = H\ket{\Psi, t} \qquad (3.39)$$

---

[9]The operators also annihilate particles, but not from the vacuum.

We solve this as[10]

$$|\Psi, s\rangle = \exp\bigl(-iH(s-t)\bigr)|\Psi, t\rangle \tag{3.40}$$

Altogether the correlator reads

$$\Delta_+(y, x) = \langle 0|\phi(\vec{y}) \exp\bigl(-i(s-t)H\bigr)\phi(\vec{x})|0\rangle.$$
$$= \int \frac{d^d\vec{p}}{(2\pi)^d \, 2e(\vec{p})} \, \exp\bigl(-ip\cdot(y-x)\bigr). \tag{3.41}$$

Space and time take different roles. Nevertheless, final answer is perfectly relativistic.

Heisenberg Picture. Heisenberg picture makes relativistic properties more manifest. Translate time-dependence of state to time-dependence of operators

$$F_{\mathrm{H}}(t) := \exp\bigl(+iH(t-t_0)\bigr) F_{\mathrm{S}}(t) \exp\bigl(-iH(t-t_0)\bigr), \tag{3.42}$$

where t_0 is the reference time slice on which quantum states are defined, say $t_0 = 0$. States are therefore time-independent in Heisenberg picture

$$F_{\mathrm{S}}(t)|\Psi, t\rangle = F_{\mathrm{S}}(t) \exp\bigl(-iH(t-t_0)\bigr)|\Psi, t_0\rangle$$
$$= \exp\bigl(-iH(t-t_0)\bigr) F_{\mathrm{H}}(t)|\Psi, t_0\rangle. \tag{3.43}$$

Field ϕ in Heisenberg picture

$$\phi(x) = \phi(\vec{x}, t) = \exp\bigl(+iHt\bigr)\phi(\vec{x})\exp\bigl(-iHt\bigr)$$
$$= \int \frac{d^d\vec{p}}{(2\pi)^d \, 2e(\vec{p})} \bigl(e^{-ip\cdot x}a(\vec{p}) + e^{+ip\cdot x}a^\dagger(\vec{p})\bigr) \tag{3.44}$$

- has complete spacetime dependence,
- is manifestly relativistic,
- no need to consider $\pi = \dot{\phi}$,
- obeys Klein–Gordon equation $(\partial^2 - m^2)\Delta_+ = 0$,
- same form as solution of Euler–Lagrange equations.

Correlator from Heisenberg picture (vacuum always the same)

$$\Delta_+(y, x) = \langle 0|\phi(y)\phi(x)|0\rangle. \tag{3.45}$$

Same result, but more immediate and relativistic derivation.

Correlator. Let us discuss the correlation function

$$\Delta_+(y, x) = \int \frac{d^d\vec{p}}{(2\pi)^d \, 2e(\vec{p})} \, \exp\bigl(-ip\cdot(y-x)\bigr). \tag{3.46}$$

---

[10]H is time-independent, hence time-translation is governed simply by $\exp(-iH\Delta t)$. Later we shall encounter a more difficult situation.

We know it in momentum space

$$\Delta_+(y, x) = \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \, \Delta_+(p) \exp\big(-ip\cdot(y-x)\big).$$
$$\Delta_+(p) = 2\pi\delta(p^2 + m^2)\theta(p_0) \tag{3.47}$$

How about position space?

- Due to translation symmetry: $\Delta_+(y-x) = \Delta_+(y,x)$.
- Due to Lorentz rotations $\Delta_+(x) = m^{d-1}F(m^2x^2)$.
- Distinguish three regions: future, past, elsewhere.

Klein–Gordon equation for F becomes

$$4rF''(r) + 2(d+1)F'(r) - F(r) = 0 \tag{3.48}$$

This is DG for Bessel functions $J_\alpha(z)$.[11] Two solutions:

$$F_\pm(r) = r^{-(d-1)/4} J_{\pm(d-1)/2}(i\sqrt{r}). \tag{3.49}$$

First, consider future with $x = (t, 0)$ where $t = \pm\sqrt{-x^2}$.[12] Substitute in above momentum space expression

$$\begin{aligned}
\Delta_+(x) &= \int \frac{d^d\vec{p}}{(2\pi)^d \, 2e(\vec{p})} \, \exp\big(-ite(\vec{p})\big). \\
&= \frac{\mathrm{Vol}(S^{d-1})}{2(2\pi)^d} \int_0^\infty \frac{dp \, p^{d-1}}{\sqrt{p^2 + m^2}} \, \exp\big(-it\sqrt{p^2 + m^2}\big). \\
&= \frac{\mathrm{Vol}(S^{d-1})}{2(2\pi)^d} \int_m^\infty de(e^2 - m^2)^{(d-2)/2} \exp(-ite) \\
&\sim e^{-imt} \qquad \text{for } t \to \pm\infty.
\end{aligned} \tag{3.50}$$

Oscillation with positive frequency m. Same for past. Fixes linear combination of $F_\pm$ (Hankel H_α) [13]

$$\Delta_+(y-x) \sim \frac{m^{(d-1)/2}}{(y-x)^{(d-1)/2}} \, H_{(d-1)/2}\big(m\sqrt{-(y-x)^2}\big). \tag{3.51}$$

For space-like separation

$$\Delta_+(x) \sim e^{-mr} \qquad \text{for } r = \sqrt{x^2} \to \infty. \tag{3.52}$$

Non-zero, but exponentially decaying (range m). Fixes linear combination of $F_\pm$ (modified Bessel K_α)

$$\Delta_+(y-x) \sim \frac{m^{(d-1)/2}}{(y-x)^{(d-1)/2}} \, K_{(d-1)/2}\big(m\sqrt{(y-x)^2}\big). \tag{3.53}$$

---

[11]Bessel functions are well-known solutions for spherical waves.
[12]Can go to such a frame with $x = (t, 0)$.
[13]The asymptotic behaviour e^{-imt} determines which of the two Hankel functions $H^{(1)}$ and $H^{(2)}$ applies to past and future.

Same as in relativistic QM. Causality?

Before we discuss causality, let us summarise in this figure:

(3.54)

Note that there are also delta-function contributions for light-like separation.

Unequal-Time Commutator. Okay to violate causality as long as not measured.

Correct question: Can one measurement at x influence the other at y? Consider commutator

$$\Delta(y - x) := [\phi(y), \phi(x)]. \tag{3.55}$$

We can relate to above correlators by inserting between vacua

$$\Delta(y - x) = \langle 0|[\phi(y), \phi(x)]|0\rangle = \Delta_+(y - x) - \Delta_+(x - y). \tag{3.56}$$

Observation:

- Δ_+ is symmetric for space-like separations,
- $\phi(x)$ and $\phi(y)$ commute for space-like separations,
- follows also from invariance and equal-time commutator,
- causality preserved.

Note: two contributions cancel for space-like separations:

- Particle created at x and annihilated at y cancels against
- particle created at y and annihilated at x.

However, commutator non-trivial for time-like separations

$$\Delta(y - x) \sim \frac{m^{(d-1)/2}}{(y - x)^{(d-1)/2}}\, J_{(d-1)/2}\big(m\sqrt{-(y - x)^2}\big). \tag{3.57}$$

Time-like separated measurements can influence each other.

Finally we can recover equal-time commutators. For two fields ϕ the commutator follows from symmetry of the integral

$$\Delta(0, \vec{y} - \vec{x}) = \int \frac{d^d\vec{p}}{(2\pi)^d\, 2e(\vec{p})} \left(e^{-i\vec{p}\cdot(\vec{y}-\vec{x})} - e^{i\vec{p}\cdot(\vec{y}-\vec{x})}\right) = 0. \tag{3.58}$$

3.11

The commutator between a field ϕ and its conjugate π yields

$$\frac{\partial}{\partial x^0}\Delta(0, \vec{y} - \vec{x}) = \frac{i}{2}\int \frac{d^d\vec{p}}{(2\pi)^d}\left(e^{-i\vec{p}\cdot(\vec{y}-\vec{x})} + e^{i\vec{p}\cdot(\vec{y}-\vec{x})}\right) = i\delta^d(\vec{x} - \vec{y}) \qquad (3.59)$$

in agreement with the fundamental commutator $[\phi(\vec{y}), \pi(\vec{x})]$.

3.5 Sources

We have a free field, we have discussed correlators of two fields, but there is not much else we can do besides adding interactions (later).

As a first step towards interactions, let us discuss driving the field by an external source $\rho(x)$

$$-\partial^2\phi(x) + m^2\phi(x) = \rho(x). \qquad (3.60)$$

The source is non-zero only for some finite interval of time.

Question: Given an initial field $\phi_i(x)$ obeying the homogeneous Klein–Gordon equation, how to determine the solution $\phi(x)$ of inhomogeneous Klein–Gordon equation such that $\phi(x) = \phi_i(x)$ for all $t < t_0$ before activation of the source at t_0? In particular, what is the final field $\phi_f(x)$ which must also obey the homogeneous Klein–Gordon equation.

$$(3.61)$$
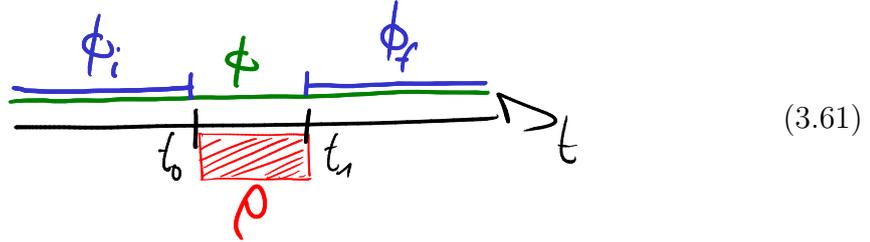
Due to linearity we make the ansatz[14]

$$\phi(x) = \phi_i(x) + \Delta\phi(x), \quad \Delta\phi(x) = \int d^{d+1}y\, G_R(x - y)\,\rho(y) \qquad (3.62)$$

where G_R is the retarded propagator or Green('s) function

$$-\partial^2 G_R(x) + m^2 G_R(x) = \delta^{d+1}(x), \qquad G_R(x) = 0 \text{ for } x^0 < 0. \qquad (3.63)$$

The first equation is conveniently solved in momentum space

$$G(x) = \int \frac{d^{d+1}p}{(2\pi)^{d+1}}\, e^{-ip\cdot x}\, G(p) \qquad (3.64)$$

with Klein–Gordon equation and solution[15]

$$p^2 G(p) + m^2 G_R(p) = 1, \qquad G(p) = \frac{1}{p^2 + m^2}. \qquad (3.65)$$

---

[14]Admittedly, this is a perfectly classical problem already encountered in electrodynamics.

[15]As such, this solution is in fact slightly ill-defined, second condition resolves the ambiguities as we shall see.

Let us see how to incorporate the second relation. Write as

$$G(x) = \int \frac{d^d\vec{p}}{(2\pi)^d} e^{-i\vec{p}\cdot\vec{x}} \int \frac{de}{2\pi} e^{-iet} \frac{-1}{e^2 - e(\vec{p})^2}$$

$$= \int \frac{d^d\vec{p}}{(2\pi)^d} e^{-i\vec{p}\cdot\vec{x}} \int \frac{de}{2\pi} \frac{-1}{2e(\vec{p})} \left(\frac{e^{-iet}}{e - e(\vec{p})} - \frac{e^{-iet}}{e + e(\vec{p})} \right). \quad (3.66)$$

Solve Fourier integrals by residue theorem in complex plane. The integral over e runs from $-\infty$ to $+\infty$; close contour! This is done by a semi-circle in the complex plane with very large radius. Contribution must vanish, consider exponent:

$$\exp(-iet) = \exp(-it\,\mathrm{Re}\,e)\exp(t\,\mathrm{Im}\,e). \quad (3.67)$$

Only second term suppresses contribution:

- For $t > 0$ we need $\mathrm{Im}\,e < 0$: Close contour in lower half.
- For $t < 0$ we need $\mathrm{Im}\,e > 0$: Close contour in upper half.
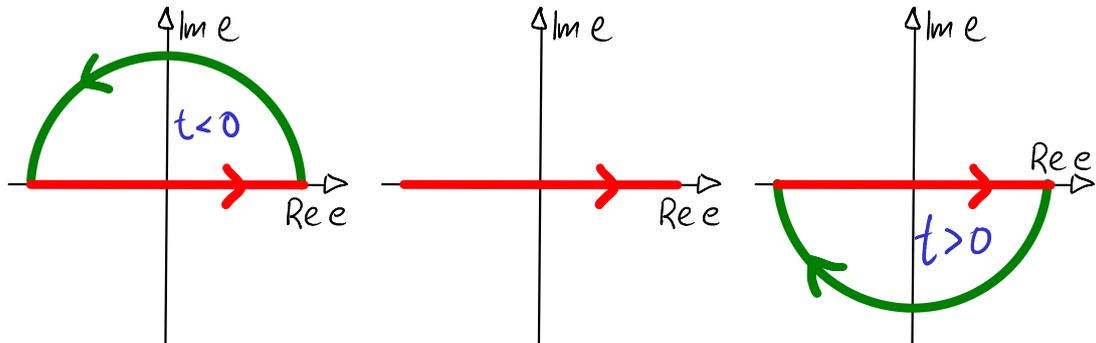

$$(3.68)$$

We have two poles at $e = \pm e(\vec{p})$ on the real axis = contour. Need to decide how they contribute to residues.

For retarded propagator, we want $G_{\mathrm{R}}(x) = 0$ for $t < 0$. Achieved by shifting poles slightly into lower half plane[16]

$$G_{\mathrm{R}}(p) = \frac{1}{p^2 + m^2 - ip_0\epsilon}. \quad (3.69)$$
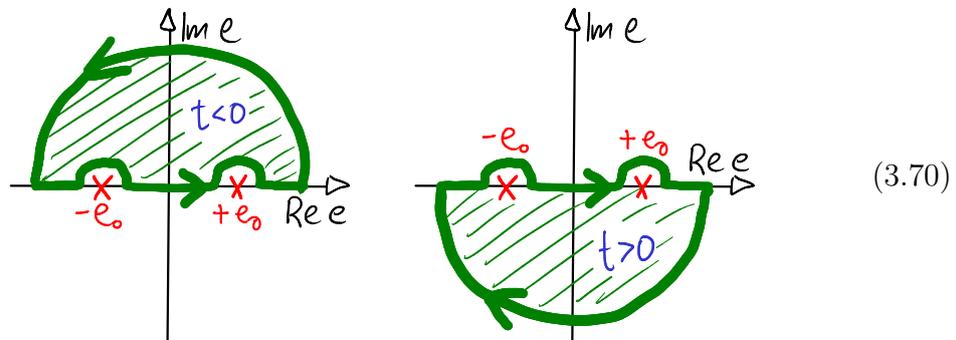
$$(3.70)$$

No poles in upper half plane, hence $G_{\mathrm{R}}(x)$ for $t < 0$. For $t > 0$, however, both

---

[16]Alternatively, deform contour to close above points.

poles contribute a residue, we obtain

$$G_{\mathrm{R}}(x) = i\theta(t) \int \frac{d^d\vec{p}}{(2\pi)^d\, 2e(\vec{p})}\left(e^{-ip\cdot x} - e^{+ip\cdot x}\right)$$

$$= i\theta(t)\big(\Delta_+(x) - \Delta_+(-x)\big) = i\theta(t)\Delta(x). \tag{3.71}$$

Nice: relation between correlation functions and propagators. Yields position space form for propagator.

Confirm that it satisfies defining relation:[17]

$$(-\partial^2 + m^2)G_{\mathrm{R}}(x) = i\theta(t)\,(-\partial^2 + m^2)\Delta(x) + \frac{\partial}{\partial t}\big(i\delta(t)\,\Delta(x)\big)$$

$$+ i\delta(t)\,\dot{\Delta}(x) = \delta^{d+1}(x). \tag{3.72}$$

- First term vanishes because Δ is on shell.
- Second term vanishes because $[\phi(\vec{x}), \phi(\vec{y})] = 0$.
- Third term uses $[\phi(\vec{x}), \phi(\vec{y})] = i\delta^d(\vec{x} - \vec{y})$.

We can now determine the contribution to ϕ from the source. Let us focus on the future after the source is switched off[18]

$$\Delta\phi(x) = i \int d^{d+1}y\, \Delta(x - y)\, \rho(y). \tag{3.73}$$

Transform this to momentum space with

$$\rho(x) = \int \frac{d^{d+1}p}{(2\pi)^{d+1}}\, e^{-ip\cdot x}\, \rho(p), \quad \rho(p)^* = \rho(-p) \tag{3.74}$$

yields

$$\Delta\phi(x) = \int \frac{d^d\vec{p}}{(2\pi)^d\, 2e(\vec{p})}\left(ie^{-ip\cdot x}\,\rho(p) - ie^{+ip\cdot x}\,\rho^*(p)\right). \tag{3.75}$$

As we know the homogeneous Klein–Gordon equation is solved by

$$\phi_{\mathrm{i}}(x) = \int \frac{d^d\vec{p}}{(2\pi)^d\, 2e(\vec{p})}\left(a(\vec{p})e^{-ip\cdot x} + a^\dagger(\vec{p})e^{ip\cdot x}\right). \tag{3.76}$$

Let this represent the solution in the distant past. Then the solution ϕ_{f} in the distant future is obtained by replacing

$$a(\vec{p}) \mapsto a(\vec{p}) + i\rho(e(\vec{p}), \vec{p}). \tag{3.77}$$

We notice that only the Fourier modes of the source ρ on the mass shell can actually drive the field ϕ.

We can now ask how much energy, momentum or particle number the source ρ transfers to the field, e.g.

$$\Delta E = \langle 0|H(t > t_1)|0\rangle - \langle 0|H(t < t_0)|0\rangle. \tag{3.78}$$

---

[17]Distribute the derivatives in a suitable way between θ and Δ.
[18]Can therefore set $\theta(y^0 - x^0) = 1$.

The contributions from the quantum modes a, $a^\dagger$ drop out. What remains is manifestly positive for $E = P_0$ and N [19]

$$\Delta P_\mu = \int \frac{d^d\vec{p}}{(2\pi)^d\, 2e(\vec{p})}\, p_\mu(\vec{p})\, |\rho(\vec{p})|^2,$$

$$\Delta N = \int \frac{d^d\vec{p}}{(2\pi)^d\, 2e(\vec{p})}\, |\rho(\vec{p})|^2. \tag{3.79}$$

---

[19] ΔN is not manifestly integer for an external field. However, if ρ is quantum, $\rho^\dagger(\vec{p})\rho(\vec{p})$ should again lead to an integer ΔN.

4 Symmetries

So far we have not discussed symmetries. QFT does not actually need symmetries, but they help very much in restricting classes of models, providing stability and simplifying calculations as well as results.[1]

For example, in most cases QFT's have some symmetry of space and time. Particularly in fundamental particle physics all models have relativistic invariance or Poincaré symmetry.

Symmetries are some transformations of the fields $\phi \to \phi'$ that map solutions of the equations of motion to other solutions. Hence they can be used to generate a whole class of solutions from a single one.

We shall discuss the action of various types of symmetries, their groups and representations, and the resulting conserved charges via Noether's theorem. Most of the discussion applies to classical and quantum field theories.

4.1 Internal Symmetries

Let us first discuss internal symmetries. In a QFT with several fields, these typically transform the fields into each other in some way without making reference to their dependence on space or time.

The simplest example is a complex scalar field $\phi(x)$ with Lagrangian and corresponding equation of motion

$$\mathcal{L} = -\partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi, \qquad \begin{array}{l} \partial^2 \phi - m^2 \phi = 0, \\ \partial^2 \phi^* - m^2 \phi^* = 0. \end{array} \qquad (4.1)$$

Consider a global transformation of the fields

$$\phi'(x) = e^{+i\alpha} \phi(x), \qquad \phi^{*\,\prime}(x) = e^{-i\alpha} \phi^*(x). \qquad (4.2)$$

It maps a solution of the equations of motion to another solution[2]

$$\partial^2 \phi' - m^2 \phi' = e^{i\alpha} \left(\partial^2 \phi - m^2 \phi \right) = 0. \qquad (4.3)$$

Moreover the symmetry leaves the Lagrangian and the action invariant

$$\mathcal{L}(\phi', \partial_\mu \phi') = \mathcal{L}(\phi, \partial_\mu \phi), \qquad S[\phi'] = S[\phi]. \qquad (4.4)$$

---

[1]For free particles symmetries are not that helpful, the true power of symmetries arises in interacting situations.

[2]The transformed field ϕ' satisfies the equation of motion because ϕ does. Same for complex conjugate field ϕ^*.

Any such transformation must be a symmetry because it maps extrema of the action to extrema and hence solutions to solutions. Symmetries of the action are more powerful than mere symmetries of the equations of motion. In the following we will only consider symmetries of the action.[3]

Noether's Theorem. Every continuous global symmetry of the action leads to a conserved current and thus a conserved charge for solutions of the equations of motion.

Let us derive the theorem: Consider a solution ϕ of the equations of motion. By construction, any variation of the Lagrangian is a total derivative[4]

$$
\begin{aligned}
\delta\mathcal{L} &= \frac{\delta\mathcal{L}}{\delta\phi}\,\delta\phi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\,\partial_\mu\delta\phi \\
&= \partial_\mu\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\,\delta\phi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\,\partial_\mu\delta\phi = \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\,\delta\phi\right).
\end{aligned}
\tag{4.5}
$$

Suppose now $\delta\phi$ is the infinitesimal field variation of a continuous symmetry. We know that $\delta S = 0$, hence the Lagrangian can only change by some total derivative

$$
\delta\mathcal{L} = \delta\alpha\,\partial_\mu J_0^\mu.
\tag{4.6}
$$

Equating the two expressions for $\delta\mathcal{L}$ we find a current[5]

$$
J^\mu = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\,\frac{\delta\phi}{\delta\alpha} - J_0^\mu
\tag{4.7}
$$

which is conserved for every solution ϕ

$$
\partial_\mu J^\mu = 0.
\tag{4.8}
$$

Furthermore, a conserved current implies a conserved charge

$$
Q(t) = \int d^d\vec{x}\, J^0(t,\vec{x})
\tag{4.9}
$$

if we assume that the field vanishes sufficiently fast at spatial infinity

$$
\dot{Q} = \int d^d\vec{x}\,\partial_0 J^0 = -\int d^d\vec{x}\,\partial_k J^k = 0.
\tag{4.10}
$$

The conserved charge actually generates an infinitesimal symmetry transformation via the Poisson brackets

$$
[Q, F] = -\frac{\delta F}{\delta\alpha}
\tag{4.11}
$$

as can be shown using its defining relations.

---

[3]For example the scaling transformation $\phi(x) \to e^\beta \phi(x)$ also maps solutions to solutions, but it rescales the Lagrangian $\mathcal{L}' = e^{2\beta}\mathcal{L}$. If one considers QFT's to be specified by their Lagrangians, then this symmetry of the equations of motion relates two different models $\mathcal{L}$ and $\mathcal{L}'$. We typically us the freedom to redefine the fields to bring the Lagrangian in some canonical form.

[4]Usually we can ignore this term, here it is relevant.

[5]Any term of the form $\partial_\nu B^{\mu\nu}$ with antisymmetric indices on $B^{\mu\nu}$ can be added to J^μ without modifying any of the following relations.

Example. Let us consider the complex scalar field. The field variation is defined by

$$\delta\phi = i\phi\,\delta\alpha, \qquad \delta\phi^* = -i\phi^*\,\delta\alpha. \tag{4.12}$$

The Lagrangian is invariant under the transformation $\delta\mathcal{L} = 0$, hence $J_0^\mu = 0$. The other term reads

$$\begin{aligned}
J^\mu &= \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\frac{\delta\phi}{\delta\alpha} + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^*)}\frac{\delta\phi^*}{\delta\alpha} \\
&= (-\partial^\mu\phi^*)(i\phi) + (-\partial^\mu\phi)(-i\phi^*) \\
&= -i(\partial^\mu\phi^*\phi - \phi^*\partial^\mu\phi).
\end{aligned} \tag{4.13}$$

The naive divergence of the current reads

$$\partial_\mu J^\mu = -i(\partial^2\phi^*\phi - \phi^*\partial^2\phi). \tag{4.14}$$

This indeed vanishes for a solution of the equations of motion.

The conserved charge reads

$$Q = i\int d^d\vec{x}\,(\dot{\phi}^*\phi - \phi^*\dot{\phi}) = i\int d^d\vec{x}\,(\pi\phi - \phi^*\pi^*). \tag{4.15}$$

Transformed to momentum space we get

$$Q = \int \frac{d^d\vec{p}}{(2\pi)^d\,2e(\vec{p})}\left(a^*(\vec{p})a(\vec{p}) - b^*(\vec{p})b(\vec{p})\right). \tag{4.16}$$

This charge is indeed time-independent and (Poisson) commutes with the Hamiltonian. As expected, it obeys

$$\{Q,\phi\} = -i\phi = -\frac{\delta\phi}{\delta\alpha}, \qquad \{Q,\phi^*\} = +i\phi^* = -\frac{\delta\phi^*}{\delta\alpha}. \tag{4.17}$$

We furthermore observe a relation to the number operators

$$Q = N_a - N_b. \tag{4.18}$$

In the quantum theory, Q therefore measures the number of particles created by $a^\dagger$ minus the number of particles created by $b^\dagger$.

Despite the similarities, there is a crucial difference to the number operator: The charge Q is associated to a symmetry, whereas a single number operator N is not.[6] In a symmetric theory with interactions, Q is conserved while N is in general not.

Quantum Action. Let us briefly state how to represent this symmetry in the quantum theory where $Q = N_a - N_b$ becomes a quantum operator. It is obviously hermitian

$$Q^\dagger = Q. \tag{4.19}$$

---

[6]One might construct a non-local symmetry transformation corresponding the number operator in a free field theory. However, this symmetry would not generalise to interactions.

It obeys the following commutation relations with creation and annihilation operators

$$[Q, a(x)] = -a(x), \qquad\qquad [Q, b(x)] = +b(x),$$
$$[Q, a^\dagger(x)] = +a^\dagger(x), \qquad\qquad [Q, b^\dagger(x)] = -b^\dagger(x). \qquad (4.20)$$

This tells us that particles of type a carry positive unit charge while the antiparticles of type b carry negative unit charge.

The commutators of spacetime fields $\phi \sim a^\dagger + b$ read

$$[Q, \phi(x)] = +\phi(x), \qquad [Q, \phi^\dagger(x)] = -\phi^\dagger(x), \qquad (4.21)$$

which tell us that ϕ and $\phi^\dagger$ carry charges $+1$ and -1, respectively. The commutators are also in agreement with the classical result that charges generate infinitesimal transformations, i.e.

$$[Q, \phi] = +\phi = -i\frac{\delta\phi}{\delta\alpha}, \qquad [Q, \phi^\dagger] = -\phi^\dagger = -i\frac{\delta\phi^\dagger}{\delta\alpha}. \qquad (4.22)$$

For finite transformations we introduce the operator

$$U(\alpha) = \exp(i\alpha Q). \qquad (4.23)$$

We can convince ourselves that it obeys the following algebra with the fields[7]

$$U(\alpha)\,\phi(x)\,U(\alpha)^{-1} = e^{+i\alpha}\phi(x) = \phi'(x),$$
$$U(\alpha)\,\phi^\dagger(x)\,U(\alpha)^{-1} = e^{-i\alpha}\phi^\dagger(x) = \phi^{\dagger\prime}(x). \qquad (4.24)$$

So $U(\alpha)$ generates a finite symmetry transformation by means of conjugation while Q generates the corresponding infinitesimal transformation by means of commutators.

Note that the operator $U(\alpha)$ is unitary because Q is hermitian

$$U(\alpha)^\dagger = \exp(-i\alpha Q^\dagger) = \exp(-i\alpha Q) = U(-\alpha) = U(\alpha)^{-1}. \qquad (4.25)$$

A crucial property of symmetries in QFT is that they are represented by *unitary* operators. This is required to make expectation values invariant under symmetry.

The symmetry group for the complex scalar is simply U(1).

The above discussions only applied to operators, let us finally discuss transformations for states. States transform under finite transformations as

$$|\Psi'\rangle = U(\alpha)|\Psi\rangle. \qquad (4.26)$$

Typically the vacuum is uncharged under symmetries[8]

$$Q|0\rangle = 0. \qquad (4.27)$$

The transformation for all other states in the Fock space then follows from the transformation of creation operators.

---

[7]Note that $Q\phi = \phi(Q+1)$ implies $\exp(i\alpha Q)\phi = \phi\exp(i\alpha(Q+1)) = e^{i\alpha}\phi\exp(i\alpha Q)$.

[8]This is not a requirement. In fact, a charged vacuum is related to spontaneous symmetry breaking and Goldstone particles, see QFT II. Note that ordering ambiguities arise in the determination of the charges, and are resolved by specifying the intended charge of the vacuum. Ordering ambiguities do not matter for commutators.

4.2 Spacetime Symmetries

Next we shall consider symmetries related to space and time. In relativistic theories these are the spatial rotations and Lorentz boosts (altogether called Lorentz symmetries) as well as spatial and temporal translations. In total they form the Poincaré group.

Translations. Let us start with simple translations in space and time

$$(x')^\mu = x^\mu - a^\mu. \tag{4.28}$$

We demand that the fields merely change by shifting the position argument

$$\phi'(x') = \phi(x). \tag{4.29}$$

In other words the new field evaluated at the new position equals the old field at the old position.[9] Explicitly,

$$\phi'(x) = \phi(x + a) \quad \text{or} \quad \delta\phi(x) = \delta a^\mu \partial_\mu \phi(x). \tag{4.30}$$

In order for translations to be a symmetry, we have to require that the Lagrangian does not explicitly depend on the position

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = 0, \qquad \mathcal{L}(\phi(x), \partial_\mu \phi(x), x) = \mathcal{L}(\phi(x), \partial_\mu \phi(x)). \tag{4.31}$$

Energy and Momentum. The Noether theorem equally applies to this situation, let us derive the associated currents and charges. The variation of the Lagrangian reads[10]

$$\delta\mathcal{L} = \delta a^\mu \left(\frac{\delta\mathcal{L}}{\delta\phi} \partial_\mu \phi + \frac{\delta\mathcal{L}}{\delta\partial_\nu \phi} \partial_\mu \partial_\nu \phi \right) = \delta a^\mu \partial_\mu (\mathcal{L}(\phi, \partial\phi)). \tag{4.32}$$

We therefore obtain a contribution $(J_0)^\mu_\nu = \delta^\mu_\nu \mathcal{L}$. All in all we obtain a vector of conserved currents $T^\mu{}_\nu$ (with $\partial_\mu T^\mu{}_\nu = 0$) where the index ν labels the $d + 1$ dimensions for shifting

$$T^\mu{}_\nu = \frac{\delta\mathcal{L}}{\delta(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}. \tag{4.33}$$

This object is called energy-momentum (or stress-energy) tensor. For a real scalar it reads

$$T^\mu{}_\nu = -\partial^\mu \phi \, \partial_\nu \phi + \tfrac{1}{2}\delta^\mu_\nu\big((\partial\phi)^2 + m^2\phi^2\big). \tag{4.34}$$

The corresponding conserved charge is the momentum vector

$$P_\mu = \int d^d\vec{x}\, T^0{}_\mu = \int d^d\vec{x} \left(\dot\phi \, \partial_\mu \phi + \tfrac{1}{2}\delta^0_\mu\big((\partial\phi)^2 + m^2\phi^2\big) \right) \tag{4.35}$$

---

[9]In other words, the transformation is active. One could also define $\phi'(x) = \phi(x')$ corresponding to a passive transformation.

[10]The derivative in the last term is meant to act on the x-dependence within the arguments $\phi(x)$ and $\partial\phi(x)$ of $\mathcal{L}$.

We recover the Hamiltonian as its time component

$$H = P_0 = \int d^d\vec{x} \left(\tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}(\vec{\partial}\phi)^2 + \tfrac{1}{2}m^2\phi^2 \right), \tag{4.36}$$

while the total spatial momentum simply reads

$$\vec{P} = \int d^d\vec{x}\, \dot{\phi}\, \vec{\partial}\phi. \tag{4.37}$$

Quantum Action. We have already encountered the quantum operators for energy and momentum. Recall that in momentum space they read

$$P_\mu = \int \frac{d^d\vec{p}}{(2\pi)^d\, 2e(\vec{p})}\, p_\mu a^\dagger(\vec{p}) a(\vec{p}). \tag{4.38}$$

Performing a quantum commutator with the field yields

$$[P_\mu, \phi(x)] = -i\partial_\mu\phi(x). \tag{4.39}$$

As before, we can introduce an operator $U(a)$ for finite shift transformations as the exponential

$$U(a) = \exp(ia^\mu P_\mu). \tag{4.40}$$

Conjugating a field with it yields the shifted field[11]

$$U(a)\phi(x)U(a)^{-1} = \exp(a^\mu\partial_\mu)\phi(x) = \phi(x+a) = \phi'(x). \tag{4.41}$$

Note that the operator $U(a)$ is unitary because P_μ is hermitian.

Lorentz Transformations. Next, consider Lorentz transformations

$$(x')^\mu = (\Lambda^{-1})^\mu{}_\nu x^\nu. \tag{4.42}$$

All upper (contravariant) indices transform according to the same rule as x^μ under Lorentz transformations, whereas lower (covariant) indices transform with the matrix Λ, just as ∂_μ does, e.g.

$$(\partial')_\mu = \Lambda^\nu{}_\mu \partial_\nu. \tag{4.43}$$

A product between a covariant and contravariant index is Lorentz invariant

$$(x')^\mu(\partial')_\mu = \Lambda^\rho{}_\mu(\Lambda^{-1})^\mu{}_\nu x^\nu \partial_\rho = x^\nu \partial_\nu. \tag{4.44}$$

The matrix Λ has the defining property that it leaves the metric $\eta_{\mu\nu}$ invariant

$$\eta'_{\mu\nu} = \eta_{\rho\sigma}\Lambda^\rho{}_\mu\Lambda^\sigma{}_\nu = \eta_{\mu\nu}. \tag{4.45}$$

---

[11]The exponentiated derivative $\exp(a^\mu\partial_\mu)\phi(x)$ generates all the terms in the Taylor expansion of $\phi(x+a)$ for small a.

We can write this relation also as

$$(\Lambda^{-1})^{\mu}{}_{\nu} = \eta_{\nu\sigma}\Lambda^{\sigma}{}_{\rho}\eta^{\rho\mu} =: \Lambda_{\nu}{}^{\mu}. \tag{4.46}$$

It implies that it makes no difference whether indices are raised or lowered before or after a Lorentz transformation. Correspondingly, scalar products between equal types of vectors are invariant.

Lorentz transformations combine spatial rotations (the matrix acts on two of the spatial dimensions)

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = \exp\begin{pmatrix} 0 & -\varphi \\ \varphi & 0 \end{pmatrix} \tag{4.47}$$

and Lorentz boosts (the matrix acts on time and one of the spatial dimensions)

$$\begin{pmatrix} \cosh\vartheta & \sinh\vartheta \\ \sinh\vartheta & \cosh\vartheta \end{pmatrix} = \exp\begin{pmatrix} 0 & \vartheta \\ \vartheta & 0 \end{pmatrix}. \tag{4.48}$$

There are also some discrete transformations which we shall discuss below. Here we restrict to proper orthochronous Lorentz transformations which form the Lie group $\mathrm{SO}^{+}(d,1)$.

We note that spatial rotations are generated by anti-symmetric matrices while Lorentz boosts are generated by symmetric matrices. Composing various such transformations in 2-dimensional subspaces of spacetime we conclude that Lorentz rotations are generated as

$$\Lambda^{\mu}{}_{\nu} = \exp(\omega)^{\mu}{}_{\nu} \tag{4.49}$$

where $\omega^{\mu}{}_{\nu}$ is a matrix satisfying

$$\omega^{k}{}_{l} = -\omega^{l}{}_{k}, \qquad \omega^{0}{}_{k} = \omega^{k}{}_{0}, \qquad \omega^{0}{}_{0} = \omega^{k}{}_{k} = 0. \tag{4.50}$$

Lowering the first index $\omega_{\mu\nu} = \eta_{\mu\rho}\omega^{\rho}{}_{\nu}$, this is equivalent to an anti-symmetric matrix

$$\omega_{\mu\nu} = -\omega_{\nu\mu}. \tag{4.51}$$

Angular Momentum.   For a (scalar) field the transformation reads

$$\phi'(x) = \phi(\Lambda x), \qquad \delta\phi = \delta\omega^{\mu}{}_{\nu}x^{\nu}\partial_{\mu}\phi. \tag{4.52}$$

Lorentz invariance of the action requires the Lagrangian to transform in the same way[12]

$$\delta\mathcal{L} = \delta\omega^{\mu}{}_{\nu}\,x^{\nu}\partial_{\mu}\mathcal{L} = \delta\omega_{\mu\nu}\,\partial^{\mu}(x^{\nu}\mathcal{L}). \tag{4.53}$$

Note that the measure $d^{d+1}x$ is Lorentz invariant. Comparing this to an explicit variation of $\mathcal{L}(\phi,\partial\phi)$ implies the relation

$$\frac{\delta\mathcal{L}}{\delta(\partial_{\mu}\phi)}\,\partial^{\nu}\phi = \frac{\delta\mathcal{L}}{\delta(\partial_{\nu}\phi)}\,\partial^{\mu}\phi. \tag{4.54}$$

---

[12]The anti-symmetry of $\omega_{\mu\nu}$ allows to pull x^{ν} past the derivative.

This relation holds whenever $\partial_\mu \phi$ appears only in the Lorentz-invariant combination $(\partial \phi)^2 = \eta^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi)$. For the energy-momentum tensor it implies symmetry in both indices when raised or lowered to the same level

$$T^{\mu\nu} = T^{\nu\mu}. \tag{4.55}$$

The currents $J^{\mu,\rho\sigma} = -J^{\mu,\sigma\rho}$ corresponding to the anti-symmetric matrix $\delta\omega_{\rho\sigma}$ can be expressed in terms of the energy momentum tensor T

$$J^{\mu,\rho\sigma} = T^{\mu\rho} x^\sigma - T^{\mu\sigma} x^\rho. \tag{4.56}$$

Conservation of $J^{\mu,\rho\sigma}$ is then guaranteed by conservation and symmetry of T

$$\partial_\mu J^{\mu,\rho\sigma} = T^{\sigma\rho} - T^{\rho\sigma} = 0. \tag{4.57}$$

The integral of J is the Lorentz angular momentum tensor

$$M^{\mu\nu} = \int d^d\vec{x}\, J^{0,\mu\nu} = \int d^d\vec{x} \left(T^{0\mu} x^\nu - T^{0\nu} x^\mu \right). \tag{4.58}$$

For a scalar field in $d = 3$ dimensional space we obtain the well-known spatial angular momentum

$$J^m = \tfrac{1}{2}\varepsilon^{mkl} M^{kl} = \int d^3\vec{x}\, \dot{\phi}\, ((\vec{\partial}\phi) \times \vec{x})^m. \tag{4.59}$$

Furthermore, the momentum for Lorentz boosts reads[13]

$$B^m = M^{0m} = \int d^3\vec{x} \left(T^{00} x^m \right) - P^m t. \tag{4.60}$$

We can also write the Lorentz generators in momentum space[14]

$$M^{\mu\nu} = i \int \frac{d^d\vec{p}}{(2\pi)^d 2e(\vec{p})} \left(p^\mu \partial^\nu a^*(\vec{p})a(\vec{p}) - p^\nu \partial^\mu a^*(\vec{p})a(\vec{p}) \right). \tag{4.61}$$

In the quantum theory, all components of the tensor $M^{\mu\nu}$ are hermitian operators. Consequently, the operators for finite transformations are unitary

$$U(\omega) = \exp(\tfrac{i}{2}\omega_{\mu\nu} M^{\mu\nu}), \qquad U(\omega)^\dagger = U(\omega)^{-1}. \tag{4.62}$$

The interesting conclusion is that we have found a unitary representation of the Poincaré group. As the latter is non-compact this representation is necessarily infinite-dimensional. Indeed, the field $\phi(x)$ and Fock space carry infinitely many degrees of freedom.

---

[13]Conservation basically implies that the motion of the centre of gravity (first term) is governed by the momentum (second term). Note that our convention uses $\vec{p} \sim -m\dot{\vec{x}}$.

[14]Its form is reminiscent of the position space form because Lorentz rotations in both spaces are practically the same

4.3 Poincaré Representations

Above we have derived Lorentz ($M^{\mu\nu}$) and momentum (P^μ) generators for relativistic transformations of a scalar field. Let us now discuss the algebraic foundations and generalisations.

Some Basic Definitions. Here are some sketches of basic definitions in group and representation theory.[15]

Group. A set G with an associative composition law $G \times G \to G$ (usually called multiplication), a unit element and inverse map $G \to G$.

Algebra. A vector space A with a bi-linear composition law $A \otimes A \to A$ (usually called multiplication).

Lie Group. A group G that is also a manifold.

Lie Algebra. An algebra g with an anti-symmetric product $[\cdot,\cdot]$ (called Lie bracket) that satisfies the Jacobi identity

$$\big[[a,b],c\big] + \big[[b,c],a\big] + \big[[c,a],b\big] = 0. \tag{4.63}$$

The tangent space of a Lie group G at the unit element is a Lie algebra.

Quantum Group, Quantum Algebra. The algebra of operators in quantum mechanics is called a quantum group or a quantum algebra. In addition to being an algebra, it has a unit element and an inverse for most elements $A^* \to A^*$. It can act as any of the above structures: It is an algebra. It has a subgroup A^*. The subgroup may contain Lie groups. A Lie algebra can be realised by the map $[a,b] = a \cdot b - b \cdot a$ which automatically satisfies the Jacobi identity.

Representation. A map $R : X \to \text{End}(V)$ from a group or an algebra X to linear operators (matrices, endomorphism) on some vector space V. The representation must reflect X's composition law by operator composition (matrix multiplication). If $a \cdot b = c$ then $R(a)R(b) = R(c)$.

Representation of a Lie Algebra. The Lie bracket must be represented by a commutator: If $[a,b] = c$ then $R(a)R(b) - R(b)R(a) = R(c)$.

Physics. The notation in physics often does not distinguish between abstract Lie algebra generators a and their representations $R(a)$, both may be denoted simply by a. Likewise the distinction between Lie brackets and commutators may be dropped (this is perfectly reasonable in a quantum algebra). Moreover the term representation is used not only for an operatorial version of algebra generators, but also for the space on which these operators act (in mathematics: module of the algebra).

Poincaré Algebra. It is straight-forward to derive the algebra of infinitesimal transformations from the operators derived earlier

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\eta^{\nu\rho}M^{\mu\sigma} - i\eta^{\mu\rho}M^{\nu\sigma} - i\eta^{\nu\sigma}M^{\mu\rho} + i\eta^{\mu\sigma}M^{\nu\rho},$$

---

[15]See a textbook for proper definitions.

$$[M^{\mu\nu}, P^\rho] = i\eta^{\nu\rho}P^\mu - i\eta^{\mu\rho}P^\nu,$$
$$[P^\mu, P^\nu] = 0. \tag{4.64}$$

These define the so-called Poincaré algebra. The operators $M^{\mu\nu}$ generate the algebra $\mathrm{so}(d,1)$ of Lorentz (orthogonal) transformations in $d+1$ spacetime dimensions. The spatial components M^{jk} generate the algebra $\mathrm{so}(d)$ of rotations in d spatial dimensions.

The Poincaré group is obtained by exponentiating the algebra

$$g(\omega, a) = \exp\left(\tfrac{i}{2}\omega_{\mu\nu}M^{\mu\nu} + ia_\mu P^\mu\right). \tag{4.65}$$

More precisely it is the component of the Poincaré group connected to the identity element. It includes $\mathrm{Spin}^+(d,1)$, the double cover of the proper orthochronous Lorentz group, along with translations.

The above algebra generators and group elements should be viewed as abstract objects in algebra without immediate connection to a physics problem. Above we have found an explicit representation (M, P) of the Poincaré algebra and the corresponding representation $U(\omega, a)$ of the Poincaré group acting on Fock space of a scalar particle. Since (M, P) commute with the number operator N, the representation is reducible.[16]

- The most relevant representation is the one acting on single particle states.
- The other representations are symmetric tensor powers of it.
- The vacuum transform in the trivial representation.
- The single-particle representation is complex, unitary, infinite-dimensional and irreducible.

Unitary Irreducible Representations. We can reproduce what we have learned about the free Klein–Gordon field from representation theory of the Poincaré algebra. We can also learn how to generalise the construction. Let us therefore investigate the unitary irreducible representations of the Poincaré group (Wigner's classification). These will be the elementary building blocks for physical theories with relativistic invariance. The derivation will parallel the derivation of unitary irreducible representations of the rotation group $\mathrm{SO}(3) \simeq \mathrm{SU}(2)$ (in QM1) which leads an understanding of spin. Here the result will characterise the types of admissible particles in a relativistic QFT.

First, we should look for commuting (combinations of) elements of the algebra. Their eigenvalues classify representations because if measured on one state, any other state related to it by symmetry must have the same eigenvalue. The analog for $\mathrm{so}(3)$ is the operator J^2. There, a representation of spin j is uniquely characterised by the eigenvalue $j(j+1)$ of J^2.

We notice that the Poincaré algebra possesses a quadratic invariant

$$P^2 = P^\mu P_\mu. \tag{4.66}$$

---

[16]In other words, the action of (M, P) neither creates nor annihilates particles and will therefore maps $\mathbb{V}_n \to \mathbb{V}_n$. The representation on Fock space thus splits into representations on the individual $\mathbb{V}_n$.

This combination obviously commutes with all the momenta P^μ. It also commutes with the Lorentz generators $M^{\mu\nu}$ because it is constructed as a scalar product.

The combination P^2 must be represented by a unique number on an irreducible representation. Otherwise one could split the representation according to their eigenvalues of P^2. Clearly, P^2 measures the mass of a particle

$$P^2 = -m^2. \tag{4.67}$$

For unitary representations P^2 must be real. There are three cases to be distinguished

- $P^2 < 0$, i.e. massive particles,
- $P^2 = 0$, i.e. massless particles,
- $P^2 > 0$, i.e. tachyons.

We shall discuss the massive case in detail and comment only briefly on the others.

The next observation is that the momentum generators P^μ span an abelian ideal[17] of the Poincaré algebra. For abelian ideals the representation space is spanned by simultaneous eigenvectors of all its elements. More concretely, the space is spanned by momentum eigenstates $|p\rangle$ with $P_\mu|p\rangle = p_\mu|p\rangle$. We have already fixed $P^2 = -m^2$ and hence we must restrict to a mass shell $p^2 = -m^2$. As the representation of finite transformations is given by $\exp(ia{\cdot}P)$, the representation is necessarily complex.

$$\tag{4.68}$$
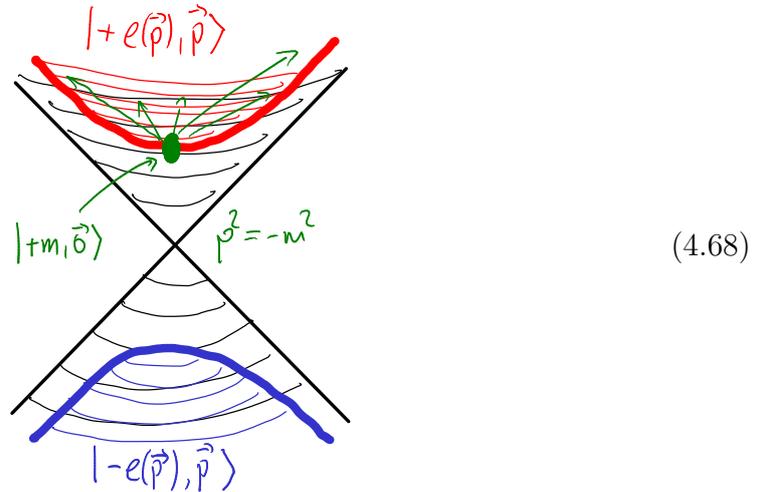
This completes the discussion of the representation of momentum generators P^μ. What about the Lorentz generators $M^{\mu\nu}$?

The condition $p^2 = -m^2$ has two connected components with positive and negative energy, respectively. Orthochronous Lorentz boosts can map between any two momentum vectors on a single mass shell. Mapping between the two mass shells is achieved only by discrete time reversal transformations which we will consider later. For an irreducible representation of the orthochronous Poincaré group, all

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[17]An ideal is a subalgebra such that brackets between its elements and elements of the algebra always end up in the subalgebra, here $[M,P] \sim P$.

energies must have the same sign. The positive-energy representation space is now spanned by the vectors $|+e_m(\vec{p}),\vec{p}\rangle$. For negative energies we can define states $|-e_m(\vec{p}),\vec{p}\rangle$ or alternatively use the hermitian conjugates $|+e_m(\vec{p}),\vec{p}\rangle^\dagger \sim \langle +e_m(\vec{p}),\vec{p}|$. Let us consider only positive energies from now on; negative energy representations are analogous. All admissible momentum vectors p can be mapped into each other by Lorentz transformations. For further discussions let us restrict to $p = (m,0)$ which is as good as any other.

Among the Lorentz generators, there are some which change $\vec{p}$. These are the Lorentz boosts. The Lorentz boosts ensure that the following discussion for $p = (m,0)$ equivalently applies to any other $\vec{p}$.

The transformations which do not change $p = (m,0)$ are the spatial rotations forming the orthogonal group $\mathrm{SO}(d)$ or its double cover $\mathrm{Spin}(d)$.[18] This group is called the little group (physics) or stabiliser (mathematics) of p. The representation subspace with fixed $\vec{p}$ must therefore transform under a representation of $\mathrm{Spin}(d)$. For the most relevant case of $d = 3$ spatial dimensions, the unitary irreducible representations of $\mathrm{Spin}(3) = \mathrm{SU}(2)$ are labelled by a non-negative half-integer j. Their representation space is spanned by $2j+1$ vectors $|-j\rangle, |-j-1\rangle, \ldots, |+j-1\rangle, |+j\rangle$ with definite z-component of spin. An equivalent representation of $\mathrm{Spin}(d)$ must apply to all momenta $\vec{p}$ because it can be shifted to the point $\vec{p} = \vec{0}$.[19] We have now considered all algebra generators and hence the representation is complete. The representation space is thus spanned by the states $|\vec{p},j_3\rangle = |\vec{p}\rangle \otimes |j_3\rangle$.

Altogether we find that the massive UIR's of the Poincaré algebra are labelled by their mass $m > 0$, the sign of energy and a unitary irreducible representation of $\mathrm{Spin}(d)$. In the case of $d = 3$, the latter UIR are labelled by a non-negative half-integer j. The representation space for $(m,\pm,j)$ is spanned by the vectors

$$|\vec{p},j_3\rangle_{(m,\pm,j)} \tag{4.69}$$

with continuous $\vec{p}$ and discrete $j_3 = -j, -j+1, \ldots, j-1, j$.

For spin $j = 0$ the representation space is simply spanned by momentum eigenstates $|\vec{p}\rangle$ with arbitrary three-momentum $\vec{p}$. These are just the single-particle states of a scalar field. The conjugate states $\langle \vec{p}|$ also transform in a UIR, but one with negative momentum.

The next interesting case is $j = \frac{1}{2}$ which we shall discuss in the following section.

In addition, there are massless representations of positive or negative energy. They are classified by a representation of $\mathrm{Spin}(d-1)$.[20] For $d = 3$ the massless representations of $\mathrm{Spin}(2) = \mathrm{U}(1)$ are labelled by a positive or negative half-integer h known as helicity. There is only one state in the representation $(0,\pm,h)$ with

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[18]Reflections extend $\mathrm{SO}(d)$ to $\mathrm{O}(d)$ or $\mathrm{Spin}(d)$ to $\mathrm{Pin}(d)$, but they are not included in the identity component of the Poincaré algebra.

[19]Each momentum vector $\vec{p}$ has a different stabiliser subgroup $\mathrm{Spin}(d) \subset \mathrm{Spin}(d,1)$, but these are all equivalent, and the same applies to their representation.

[20]In fact, the stabiliser is the euclidean group in $d-1$ dimensions which also allows for so-called continuous spin representations.

given helicity

$$|\vec{p}\rangle_{(0,\pm,h)}. \tag{4.70}$$

Last but not least, there is the trivial representation with $P = 0$. The Fock space vacuum transforms under it. Finally, there are tachyonic representations with $P^2 > 0$, but the latter are typically non-unitary.

4.4 Discrete Symmetries

In addition to the continuous symmetries discussed above, there are also relevant discrete symmetries. The most prominent ones are parity, time reversal and charge conjugation. Let us discuss them for the example of a complex scalar field.

Parity. Spatial rotations in d dimensions form the special orthogonal group $\mathrm{SO}(d)$. However, also spatial reflections preserve all distances, and it is natural to consider them among the symmetries, too. Reflections were long believed to be a symmetry of nature, until the electroweak interactions were shown to violate parity symmetry. On the mathematical side, reflections flip the orientation and together with the rotations they form the general orthogonal group $\mathrm{O}(d)$.[21]

For an odd number of spatial dimensions d, it is convenient to introduce parity P as the transformation which inverts all spatial components of the position vector

$$P : (t, \vec{x}) \mapsto (t, -\vec{x}). \tag{4.71}$$

It is an element of $\mathrm{O}(d)$, but not of $\mathrm{SO}(d)$, and it is convenient to choose this element because it does not introduce any preferred directions. There are many more orientation-inverting elements in $\mathrm{O}(d)$; these can be obtained as products of P with elements of $\mathrm{SO}(d)$. Hence it is sufficient to consider only P. In spacetime, introducing parity enlarges the identity component of the Lorentz group $\mathrm{SO}^+(d,1)$ to the orthochronous Lorentz group $\mathrm{O}^+(d,1)$.

A scalar field should transform under parity as follows

$$P\phi(t, \vec{x})P^{-1} = \eta_{\mathrm{P}}\phi(t, -\vec{x}),$$
$$P\phi^\dagger(t, \vec{x})P^{-1} = \eta_{\mathrm{P}}^*\phi^\dagger(t, -\vec{x}), \tag{4.72}$$

where the constant η_{P} is the intrinsic parity of the field ϕ. We want that two parity transformations equal the identity $P^2 = 1$, therefore the parity can be either positive or negative, $\eta_{\mathrm{P}} = \pm 1$.

For the creation and annihilation operators it implies a transformation which reverses the momentum

$$Pa(\vec{p})P^{-1} = \eta_{\mathrm{P}}a(-\vec{p}), \qquad\qquad Pa^\dagger(\vec{p})P^{-1} = \eta_{\mathrm{P}}a^\dagger(-\vec{p}),$$
$$Pb(\vec{p})P^{-1} = \eta_{\mathrm{P}}b(-\vec{p}), \qquad\qquad Pb^\dagger(\vec{p})P^{-1} = \eta_{\mathrm{P}}b^\dagger(-\vec{p}). \tag{4.73}$$

It is a unitary operation.

---

[21]The double cover of $\mathrm{O}(d)$ is called $\mathrm{Pin}(d)$ in analogy to $\mathrm{Spin}(d)$ which is the double cover of $\mathrm{SO}(d)$.

Time Reversal. The other discrete transformation of the Lorentz group is time reversal

$$T : (t, \vec{x}) \mapsto (-t, \vec{x}). \tag{4.74}$$

It enlarges the orthochronous Lorentz group $\mathrm{O}^+(d, 1)$ to the complete Lorentz group $\mathrm{O}(d, 1)$.

Time reversal is a rather special transformation due to the distinguished role of time in quantum mechanics and special relativity.

For a field ϕ we expect

$$\begin{aligned}
T\phi(t, \vec{x})T^{-1} &= \eta_{\mathrm{T}}\phi(-t, \vec{x}), \\
T\phi^\dagger(t, \vec{x})T^{-1} &= \eta_{\mathrm{T}}^*\phi^\dagger(-t, \vec{x}).
\end{aligned} \tag{4.75}$$

Comparing to the mode expansion of fields, this could be implemented by a linear transformation of the type $a^\dagger(\vec{p}) \mapsto b(-\vec{p})$. However, such a transformation would not act well on Fock space because it would annihilate all states (but the vacuum).[22] Instead, time reversal (sometimes called motion reversal) is defined by an anti-linear operator which also conjugates plain complex numbers, let us denote it by $\bar{T}$. This inverts the plane wave factors $e^{\pm i p \cdot x}$ and allows to map $a^\dagger \mapsto a^\dagger$, more explicitly

$$\begin{aligned}
\bar{T}a(\vec{p})\bar{T}^{-1} &= \eta_{\bar{\mathrm{T}}}^*a(-\vec{p}), & \bar{T}a^\dagger(\vec{p})\bar{T}^{-1} &= \eta_{\bar{\mathrm{T}}}a^\dagger(-\vec{p}), \\
\bar{T}b(\vec{p})\bar{T}^{-1} &= \eta_{\bar{\mathrm{T}}}b(-\vec{p}), & \bar{T}b^\dagger(\vec{p})\bar{T}^{-1} &= \eta_{\bar{\mathrm{T}}}^*b^\dagger(-\vec{p}).
\end{aligned} \tag{4.76}$$

The difference w.r.t. parity is merely the anti-linear feature of $\bar{T}$. Time reversal actually allows for a complex $\eta_{\bar{\mathrm{T}}}$ only restricted by $|\eta_{\bar{\mathrm{T}}}|^2 = 1$.

Charge Conjugation. Also the internal symmetry groups can come along with several connected components. For example the complex scalar field has a global $\mathrm{U}(1) = \mathrm{SO}(2)$ symmetry. This can be extended to $\mathrm{O}(2)$ by adding a charge conjugation symmetry.

We already know that the complex conjugate scalar field ϕ^* or $\phi^\dagger$ satisfies the same equations of motion as the original field ϕ. Charge conjugation symmetry thus maps between the fields ϕ and $\phi^\dagger$

$$\begin{aligned}
C\phi(x)C^{-1} &= \eta_{\mathrm{C}}\phi^\dagger(x), \\
C\phi^\dagger(x)C^{-1} &= \eta_{\mathrm{C}}^*\phi(x).
\end{aligned} \tag{4.77}$$

Requiring that two charge conjugations square to unity, the parity η_{C} must be on the complex unit circle $|\eta_{\mathrm{C}}|^2 = 1$.

$$\begin{aligned}
Ca(\vec{p})C^{-1} &= \eta_{\mathrm{C}}^*b(\vec{p}), & Ca^\dagger(\vec{p})C^{-1} &= \eta_{\mathrm{C}}b^\dagger(\vec{p}), \\
Cb(\vec{p})C^{-1} &= \eta_{\mathrm{C}}a(\vec{p}), & Cb^\dagger(\vec{p})C^{-1} &= \eta_{\mathrm{C}}^*a^\dagger(\vec{p}).
\end{aligned} \tag{4.78}$$

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[22]Unless the vacuum is mapped to a different states, e.g. $\langle 0|$, which makes this definition similar to the conventional one.

Although C maps $\phi \mapsto \phi^\dagger$, it is a perfectly linear map. Charge conjugation is not complex conjugation. One might as well make an anti-linear ansatz for C, but it would lead to a transformation of the kind $a^\dagger \mapsto a$ which would again annihilate almost all of Fock space.

There are several conceptual difficulties with charge conjugation parity:

- In the presence of a corresponding internal symmetry the parity η_C actually does not have deeper meaning. In this case one can define a new charge conjugation operation C' by conjugating C by the internal symmetry. This would lead to a different η_C, and it makes sense to choose C' such that $\eta_C = 1$.
- Even if there is no continuous internal symmetry, there can be discrete internal symmetries. For example, a possible transformation for a real scalar field is $\phi \mapsto -\phi$.
- In a model with multiple fields, several independent internal parities can coexist, and there may not be a distinguished charge conjugation symmetry. In general, one would expect C to invert all internal charges.
- In the presence of some internal parity C, the spacetime parities P and T become somewhat ambiguous, as one could define $P' = PC$.[23]

Hence the choice of discrete symmetries C, P, T can be ambiguous, and its sometimes tricky to identify the most suitable (or established) one.[24]

Implications. A discrete transformation is a symmetry if it commutes with the Hamiltonian. It is natural to assume parity and time reversal as symmetries of relativistic QFT models and of nature. For a scalar field, it appears impossible to violate parity or time reversal. However, as we shall see, this need not be so for other types of fields. In nature, indeed, some of these symmetries are violated.

Discrete symmetries also lead to conserved charges in the quantum theory. States can be classified by their eigenvalue (parity) under the discrete symmetry. Typically these parities are not additive (as the electrical charge, e.g.), but they only take finitely many values (e.g. $+1$ or -1).

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[23]This may appear strange at first sight as P' would conjugate the field ϕ. But by writing $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$ we get two fields with opposite parities η_P.

[24]For example, the question of whether a certain discrete symmetry applies should be interpreted as the question where there exist *some choice* of this discrete transformation that is a symmetry.

5 Free Spinor Field

We have seen that next to the scalar field there exist massive representations of Poincaré algebra with spin. The next higher case is spin $j = \frac{1}{2}$. It is described by the Dirac equation, and as a field with half-integer spin it should obey Fermi statistics.

5.1 Dirac Equation and Clifford Algebra

Dirac Equation. Dirac attempted to overcome some of the problems of relativistic quantum mechanics by introducing a first-order wave equation.[1]

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0. \tag{5.1}$$

Here the γ^μ are some suitably chosen operators acting on ψ. This wave equation can be viewed as a factorisation of the second-order Klein–Gordon equation as follows:

$$(i\gamma^\nu \partial_\nu + m)(i\gamma^\mu \partial_\mu - m)\psi = (-\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu - m^2)\psi = 0. \tag{5.2}$$

The latter form becomes the Klein–Gordon equation provided that the γ's satisfy the Clifford algebra[2] [3]

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = -2\eta^{\mu\nu}. \tag{5.3}$$

This means that every solution of the Dirac equation also satisfies the Klein–Gordon equation and thus describes a particle of mass m.

Clifford Algebra. The Clifford algebra obviously cannot be realised in terms of plain numbers, but finite-dimensional matrices suffice. The realisation of the Clifford algebra strongly depends on the dimension and signature of spacetime.

The simplest non-trivial case is three-dimensional space (without time). A representation of the corresponding Clifford algebra is given by the 2×2 Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.4}$$

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[1] The combination of a gamma matrix and an ordinary vector $\gamma^\mu B_\mu$ is often denoted by a slashed vector $\slashed{B}$.

[2] The indices of the two derivatives are automatically symmetric, hence only the symmetrisation of $\gamma^\mu \gamma^\nu$ must equal $-\eta^{\mu\nu}$.

[3] We will use conventional γ matrices for signature $+---$ and the minus sign in the Clifford algebra adjusts for our choice of opposite signature. Alternatively, one could multiply all γ-matrices by i and drop the minus sign.

One can convince oneself that these matrices obey the algebra[4]

$$\sigma^j \sigma^k = \delta^{jk} + i\varepsilon^{jkl}\sigma^l. \tag{5.5}$$

This also implies the three-dimensional Clifford algebra

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij}. \tag{5.6}$$

In this course we will be predominantly be interested in the case of $d = 3$ spatial dimensions plus time, i.e. spacetime with $D = d + 1 = 4$ dimensions. There, the smallest non-trivial representation of the Clifford algebra is four-dimensional (coincidence!). The elements of this four-dimensional vector space are called spinors, more precisely, Dirac spinors or 4-spinors.

There are many equivalent ways to write this representation as 4×4 matrices. The best-known ones are the Dirac, Weyl and Majorana representations. These are often presented in the form of 2×2 matrices whose elements are again 2×2 matrices. The latter are written using the Dirac matrices. We shall mainly use the Weyl representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^k = \begin{pmatrix} 0 & +\sigma^k \\ -\sigma^k & 0 \end{pmatrix}. \tag{5.7}$$

One can easily confirm that these matrices obey the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$ by means of the three-dimensional Clifford algebra. A useful property of the Weyl representation is that all four gamma matrices are block off-diagonal. The Dirac and Majorana representations have different useful properties. In most situations, it is however convenient not to use any of the explicit representations, but work directly with the abstract Clifford algebra.

Solutions.   The Dirac equation is homogeneous therefore it is conveniently solved by Fourier transformation

$$\psi(x) = \int d^4p\, e^{-ip\cdot x}\psi(p), \qquad (p_\mu\gamma^\mu - m)\psi = 0. \tag{5.8}$$

To construct the solutions, let us introduce the matrices

$$\Pi^\pm = \frac{1}{2m}(m \pm p\cdot\gamma) \tag{5.9}$$

such that the Dirac equation becomes $\Pi^-\psi = 0$. We are interested in the kernel of Π^-.

As noted above, we have the identity

$$\Pi^+\Pi^-\psi = \frac{1}{4m^2}(m^2 + p^2)\psi. \tag{5.10}$$

---

[4]We will not introduce a distinguished symbol for unit matrices. 1 is the unit element. Here the term δ^{ij} has an implicit 2×2 unit matrix.

This operators acts identically on all components of ψ. Any solution therefore requires the mass shell condition $p^2 = -m^2$.

On the mass shell $p^2 = -m^2$, the operators $\Pi^\pm$ act as orthogonal projectors:

$$\Pi^\pm \Pi^\pm = \Pi^\pm, \qquad \Pi^\pm \Pi^\mp = 0. \tag{5.11}$$

Now the operators $\Pi^\pm$ are very similar.[5] Evidently, their kernels have the same dimension. Therefore $\Pi^\pm$ both have half-maximal rank. The Dirac equation therefore has two solutions for each on-shell momentum p.

A basis of two positive-energy solutions is denoted by

$$u_a(\vec{p}), \quad a = \pm, \qquad (p{\cdot}\gamma - m)u_a(\vec{p}) = 0. \tag{5.12}$$

Instead of introducing negative-energy solutions, we prefer to consider equivalent positive-energy solutions of the opposite Dirac equation $\Pi^+\psi = 0$

$$v_a(\vec{p}), \quad a = \pm, \qquad (p{\cdot}\gamma + m)v_a(\vec{p}) = 0. \tag{5.13}$$

To write such solutions explicitly, we can recycle the projectors $\Pi^\pm$ and set

$$u = \Pi^+\lambda, \qquad v = \Pi^-\lambda, \tag{5.14}$$

where λ is some spinor. The properties of the projectors immediately show that u and v are solutions to their respective equations. Note, however, that some components of λ are projected out in u and in v.

Let us consider explicitly solutions in the Weyl representation. E.g. setting $\lambda = (\kappa, 0)$ with κ some 2-spinor, we find

$$\begin{aligned}
u(\vec{p}) &= \frac{1}{2m} \begin{pmatrix} m & e(\vec{p}) + \vec{p}{\cdot}\vec{\sigma} \\ e(\vec{p}) - \vec{p}{\cdot}\vec{\sigma} & m \end{pmatrix} \begin{pmatrix} \kappa \\ 0 \end{pmatrix} \\
&= \frac{1}{2m} \begin{pmatrix} m\kappa \\ e(\vec{p})\kappa - \vec{p}{\cdot}\vec{\sigma}\kappa \end{pmatrix}, \\
v(\vec{p}) &= \frac{1}{2m} \begin{pmatrix} m\kappa \\ -e(\vec{p})\kappa + \vec{p}{\cdot}\vec{\sigma}\kappa \end{pmatrix}.
\end{aligned} \tag{5.15}$$

There are two independent choices for the 2-spinor κ, hence there are two solutions for u and v, respectively. One typically considers $u_\gamma(\vec{p})$, $v_\gamma(\vec{p})$, $\gamma = \pm$, as two pairs of fixed basis vectors for each momentum $\vec{p}$.

Altogether the general solution can now be expanded as

$$\psi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 \, 2e(\vec{p})} \left(e^{-ip{\cdot}x} u_\gamma(\vec{p}) b_\gamma(\vec{p}) + e^{ip{\cdot}x} v_\gamma(\vec{p}) a_\gamma^\dagger(\vec{p}) \right) \tag{5.16}$$

Here the negative-energy coefficient b_γ is chosen differently from a_γ because gamma matrices are generally complex and therefore also the Dirac spinor ψ.

---

[5]We might as well have declared $(i\gamma^\mu \partial_\mu + m)\psi = 0$ to be the Dirac equation.

5.2 Poincaré Symmetry

The Dirac equation is a relativistic wave equation. Translational invariant is evident. Most importantly, we have not yet shown its Lorentz covariance (although the resulting Klein–Gordon equation certainly is covariant).

Lorentz Symmetry. Let us therefore consider a Lorentz transformation $x' = \Lambda^{-1}x$ with $\Lambda(\omega) = \exp(\omega)$. Suppose ψ is a solution of the Dirac equation. It is not sufficient to use the transformation rule for scalar fields $\psi'(x') = \psi(x)$. In analogy to vectors we should also transform spinors. We make the ansatz

$$\psi'(x') = S(\omega)\psi(x), \tag{5.17}$$

where $S(\omega)$ is a matrix that acts on Dirac spinors. We then substitute $\psi'(x) = S\psi(\Lambda x)$ into the Dirac equation

$$
\begin{aligned}
0 &= i\gamma^\mu \partial_\mu \psi' - m\psi' \\
&= \big(i\gamma^\nu S\Lambda^\mu{}_\nu \partial_\mu \psi - Sm\psi\big)(\Lambda x) \\
&= S\big(iS^{-1}\gamma^\nu S\Lambda^\mu{}_\nu \partial_\mu \psi - i\gamma^\mu \partial_\mu \psi\big)(\Lambda x) \\
&= iS\big(\Lambda^\mu{}_\nu S^{-1}\gamma^\nu S - \gamma^\mu\big)(\partial_\mu \psi)(\Lambda x).
\end{aligned}
\tag{5.18}
$$

So the term in the bracket must vanish for invariance of the Dirac equation.

Indeed, the canonical Lorentz transformation of gamma matrices

$$\gamma'^\mu = (\Lambda^{-1})^\mu{}_\nu \, S\gamma^\nu S^{-1}, \tag{5.19}$$

where not only the vector index is transformed by Λ^{-1}, but also the spinor matrix is conjugated by the corresponding spinor transformation S.[6] In analogy to the invariance of the Minkowski metric, $\eta' = \eta$, the Dirac equation is invariant if the gamma matrices are invariant

$$\gamma'^\mu = \gamma^\mu. \tag{5.20}$$

This condition relates S to the Lorentz transformation Λ.

The infinitesimal form of the invariance condition reads

$$[\delta S, \gamma^\mu] - \delta\omega^\mu{}_\nu \gamma^\nu = 0, \tag{5.21}$$

Now δS does not carry any vector indices, but it should be proportional to two $\delta\omega_{\mu\nu}$ which carries two of them. We can only contract them to two gamma matrices, and we make the ansatz $\delta S = \frac{1}{2}\alpha\delta\omega_{\mu\nu}\gamma^\mu\gamma^\nu$. Substituting this into the invariance condition and using

$$[\gamma^\rho\gamma^\sigma, \gamma^\mu] = \gamma^\rho\{\gamma^\sigma, \gamma^\mu\} - \{\gamma^\rho, \gamma^\mu\}\gamma^\sigma. \tag{5.22}$$

we arrive at $(2\alpha - 1)\delta\omega^\mu{}_\nu\gamma^\nu = 0$. We conclude that a Lorentz transformation for spinors is given by the matrix

$$\delta S = \tfrac{1}{4}\delta\omega_{\mu\nu}\gamma^\mu\gamma^\nu \quad \text{or} \quad S(\omega) = \exp\big(\tfrac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu\big). \tag{5.23}$$

---

[6]In general, spinors are transformed by S from the left, co-spinors by the S^{-1} from the right.

Comparing this result to the abstract form of finite Lorentz transformations as $U(\omega) = \exp(\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu})$ we have derived a new representation on spinors

$$M^{\mu\nu} = -\tfrac{i}{4}[\gamma^\mu, \gamma^\nu]. \tag{5.24}$$

This representation obeys the Lorentz algebra derived above, i.e.
$[M^{\mu\nu}, M^{\rho\sigma}] = iM + \ldots.$

Double Cover. Spinor representations exist only for the double cover $\mathrm{Spin}(X)$ of an orthogonal group $\mathrm{SO}(X)$. Let us observe this fact in a simple example.

Consider a rotation in the x-y-plane with angle $\omega_{12} = -\omega_{21} = \varphi$. The associated finite Lorentz transformation matrix in the x-y-plane reads

$$\Lambda(\varphi) = \exp(\omega) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}. \tag{5.25}$$

The associated spinor transformation reads

$$S(\varphi) = \mathrm{diag}(e^{-i\varphi/2}, e^{+i\varphi/2}, e^{-i\varphi/2}, e^{+i\varphi/2}). \tag{5.26}$$

The vector rotation $\Lambda(\varphi)$ is 2π-periodic in φ whereas the spinor rotation is merely 4π-periodic. The rotation by $\varphi = 2\pi$ is represented by the unit matrix for vectors, but for spinors it is the negative unit matrix

$$\Lambda(2\pi) = 1 = (-1)^F, \qquad S(2\pi) = -1 = (-1)^F. \tag{5.27}$$

The spin group thus has an element which represents a rotation by 2π (irrespectively of the direction). On vector representations (integer spin) it acts as the identity, on spinor representations (half-integer spin) it acts as -1. Due to the relation between spin and statistics, the extra element is equivalent to $(-1)^F$ where F measures the number of fermions (odd for spinors, even for vectors).

Chiral Representation. There is an important feature of the spin representation $M^{\mu\nu}$ which is best observed in the Weyl representation of gamma matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}. \tag{5.28}$$

Here we have introduced the sigma matrices σ^μ, $\bar\sigma^\mu$ as an extension of the Pauli matrices σ^k to four spacetime dimensions as follows

$$\sigma^0 = \bar\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \bar\sigma^k = -\sigma^k. \tag{5.29}$$

The Lorentz representation now reads

$$M^{\mu\nu} = -\frac{i}{4}\begin{pmatrix} \sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu & 0 \\ 0 & \bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu \end{pmatrix}. \tag{5.30}$$

This representation has block-diagonal form and therefore reduces to two independent representations $M^{\mu\nu} = \mathrm{diag}(M_{\mathrm{L}}^{\mu\nu}, M_{\mathrm{R}}^{\mu\nu})$ with

$$M_{\mathrm{L}}^{\mu\nu} = -\tfrac{i}{4}(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu), \qquad M_{\mathrm{R}}^{\mu\nu} = -\tfrac{i}{4}(\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu). \qquad (5.31)$$

In other words, the Dirac spinor $\psi = (\psi_{\mathrm{L}}, \psi_{\mathrm{R}})$ transforms in the direct sum of two (irreducible) representations of the Lorentz group. The 2-spinors ψ_{L} and ψ_{R} are called left-chiral and right-chiral spinors. The massive Dirac equation, however, mixes these two representations

$$i\sigma^\mu \partial_\mu \psi_{\mathrm{R}} - m\psi_{\mathrm{L}} = 0,$$
$$i\bar{\sigma}^\mu \partial_\mu \psi_{\mathrm{L}} - m\psi_{\mathrm{R}} = 0. \qquad (5.32)$$

It is therefore convenient to use Dirac spinors for massive spinor particles. We shall discuss the massless case later on.

The decomposition into chiral parts is not just valid in the Weyl representation of the Clifford algebra. More abstractly, it is due to the existence of the matrix

$$\gamma^5 = \tfrac{i}{24}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma = i\gamma^0 \gamma^1 \gamma^2 \gamma^3. \qquad (5.33)$$

In the Weyl representation it reads $\gamma^5 = \mathrm{diag}(-1, +1)$, it therefore measures the chirality of spinors. In general, it anti-commutes with all the other gamma matrices,

$$\{\gamma^5, \gamma^\mu\} = 0. \qquad (5.34)$$

This property implies that a single gamma matrix maps between opposite chiralities, i.e. it inverts chirality. The property is also sufficient to prove commutation with $M^{\mu\nu}$. Alternatively, it follows by construction of γ^5 as a (pseudo)-scalar combination of gamma matrices.

A further useful property is

$$\gamma^5 \gamma^5 = 1, \qquad (5.35)$$

It can be used to show that the combinations $\tfrac{1}{2}(1 \pm \gamma_5)$ are two orthogonal projectors to the chiral subspaces.

Sigma Matrices.   Let us briefly discuss the sigma matrices which are chiral analogs of the gamma matrices. The sigma matrices obey an algebra reminiscent of the Clifford algebra[7][8]

$$\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = -2\eta^{\mu\nu} = \bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu. \qquad (5.36)$$

Inspection shows that all sigma matrices are hermitian

$$(\sigma^\mu)^\dagger = \sigma^\mu, \qquad (\bar{\sigma}^\mu)^\dagger = \bar{\sigma}^\mu. \qquad (5.37)$$

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[7]The assignment of bars enables a 2-dimensional representation for this algebra unlike the Clifford algebra which requires a larger 4-dimensional representation.

[8]Note that for any reasonable product of sigma matrices the sequence of factors will alternate between σ and $\bar{\sigma}$. This agrees with the fact that a single γ maps between the two chiralities.

In other words, the four sigma matrices form a real basis for 2×2 hermitian matrices. Likewise one can confirm that that the matrices $M_{\mathrm{L,R}}^{\mu\nu}$ form a real basis of 2×2 complex traceless matrices.[9] Furthermore,

$$(M_{\mathrm{L}}^{\mu\nu})^{\dagger} = -M_{\mathrm{R}}^{\mu\nu}. \tag{5.38}$$

These are just the defining relations for the fundamental representation of $\mathrm{sl}(2, \mathbb{C})$ along with its conjugate representation.[10]

The Lorentz algebra $\mathrm{so}(3, 1)$ is indeed equivalent to the algebra $\mathrm{sl}(2, \mathbb{C})$. At the level of groups, $\mathrm{Spin}^{+}(3, 1) = \mathrm{SL}(2, \mathbb{C})$ is the double cover of $\mathrm{SO}^{+}(3, 1)$.

- A chiral 2-spinor of $\mathrm{Spin}^{+}(3, 1)$ is equivalent to the fundamental representation (space) of $\mathrm{SL}(2, \mathbb{C})$.
- Similarly, a 2-spinor of opposite chirality is equivalent to the conjugate fundamental representation (space) of $\mathrm{SL}(2, \mathbb{C})$.
- Spinor representation exist only for the double-cover group $\mathrm{Spin}^{+}(3, 1)$, but not for the original Lorentz group $\mathrm{SO}^{+}(3, 1)$.

5.3 Discrete Symmetries

In addition to the continuous Poincaré symmetry and an obvious $\mathrm{U}(1)$ internal symmetry, there are several discrete symmetries and transformations which we shall now discuss. These are also needed to formulate a Lagrangian.

Parity. Spatial parity $\vec{x}' = -\vec{x}$ is the simplest discrete symmetry. We make the usual ansatz

$$\psi'(t, -\vec{x}) = \gamma_{\mathrm{P}} \psi(t, \vec{x}), \tag{5.39}$$

where γ_{P} is a matrix that induces the reflection on spinors.

The new field obeys the same old Dirac equation provided that the gamma matrices are invariant

$$\gamma'^{\mu} := \Lambda^{\mu}{}_{\nu} \gamma_{\mathrm{P}} \gamma^{\nu} \gamma_{\mathrm{P}}^{-1} \overset{!}{=} \gamma^{\mu}. \tag{5.40}$$

We need to find a matrix γ_{P} that

- commutes with γ^0 (because $\Lambda^0{}_0 = 1$),
- anti-commutes with γ^k (to compensate $\Lambda^k{}_k = -1$),
- squares to unity (because $P^2 = 1$).

This matrix is easily identified as

$$\gamma_{\mathrm{P}} = \gamma^0. \tag{5.41}$$

Note that γ_{P} interchanges the two chiralities. Hence the Dirac spinor is

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[9] A traceless 2×2 matrix has $2 \cdot 2 - 1 = 3$ degrees of freedom. If the latter are complex, there are altogether 6 real d.o.f.. A pair of anti-symmetric vector indices provides the same number of d.o.f., noting that all $M_{\mathrm{L,R}}^{\mu\nu}$ are linearly independent (over the real numbers).

[10] The group $\mathrm{SL}(N)$ of matrices with unit determinant is generated by the algebra $\mathrm{sl}(N)$ of traceless matrices.

- reducible under proper orthochronous Lorentz rotations,
- but irreducible under orthochronous Lorentz rotations.

Time Reversal. Anti-linear time (motion) reversal also has a representation on spinors

$$\psi'(-t, \vec{x}) = \gamma_{\bar{\mathrm{T}}}\psi(t, \vec{x}),\tag{5.42}$$

The anti-linear nature of $\bar{T}$ implies that a solutions of the Dirac equation should be mapped to a solution of the complex conjugated Dirac equation. In the Weyl representation this is achieved by the matrix

$$\gamma_{\bar{\mathrm{T}}} = \gamma^1\gamma^3.\tag{5.43}$$

The gamma matrices satisfy the following identity with the time reversal matrix

$$\Lambda^\mu{}_\nu\gamma_{\bar{\mathrm{T}}}(\gamma^\nu)^*\gamma_{\bar{\mathrm{T}}}^{-1} = -\gamma^\mu.\tag{5.44}$$

Charge Conjugation. The Dirac field is charged, it therefore makes sense to define charge conjugation. We will use it later to investigate the statistics associated to spinor fields.

Linear charge conjugation maps a field to its conjugate field[11]

$$\psi'(x) = \gamma_{\mathrm{C}}\psi^{\dagger\mathrm{T}}\tag{5.45}$$

such that ψ' solves the same wave equation as ψ. Let us substitute

$$(i\gamma^\mu\partial_\mu - m)\psi' = (i\gamma^\mu\partial_\mu - m)\gamma_{\mathrm{C}}\psi^{\dagger\mathrm{T}} = \big((-i(\gamma^\mu)^*\partial_\mu - m)\gamma_{\mathrm{C}}^*\psi\big)^{\dagger\mathrm{T}}.\tag{5.46}$$

This vanishes if

$$\gamma_{\mathrm{C}}(\gamma^\mu)^*\gamma_{\mathrm{C}}^{-1} = -\gamma^\mu.\tag{5.47}$$

In the Weyl representation only γ^2 is imaginary, and the condition is solved by the matrix

$$\gamma_{\mathrm{C}} = -i\gamma^2.\tag{5.48}$$

CPT-Transformation In QFT a discrete transformation of fundamental importance is the combination of charge conjugation, parity and time reversal, called CPT. Effectively, it flips the sign of all coordinates[12] and performs a complex conjugation.

A spinor transforms according to

$$\psi'(x) = \gamma_{\bar{\mathrm{T}}}\gamma_{\mathrm{P}}\gamma_{\mathrm{C}}\psi^{\dagger\mathrm{T}}(-x)\tag{5.49}$$

---

[11]The composition of adjoint and transpose operations is almost the same as complex conjugation. There is however a slight difference which becomes relevant only later.

[12]This is an orientation-preserving transformation which belongs to $\mathrm{Spin}(3,1)$, but not to $\mathrm{Spin}^+(3,1)$.

We find that the combination of matrices is just the additional gamma matrix γ^5

$$\gamma_{\bar{T}}\gamma_P\gamma_C = -i\gamma^1\gamma^3\gamma^0\gamma^2 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5. \tag{5.50}$$

This anti-commutes with all gamma matrices

$$\gamma^5\gamma^\mu\gamma^5 = -\gamma^\mu. \tag{5.51}$$

The sign is compensated by flipping the sign of all vectors.

The CPT-theorem states that all reasonable relativistic QFT's must be invariant under the CPT-transformation. They need not be invariant under any of the individual transformations.

Hermitian Conjugation. The Dirac spinor ψ is complex. To construct real quantities for use in the Lagrangian or the Hamiltonian one typically uses hermitian conjugation. However, the various gamma matrices transform differently under this operation.

The transformation can be uniformised by conjugation with some other matrix $\gamma_\dagger$

$$\gamma_\dagger(\gamma^\mu)^\dagger\gamma_\dagger^{-1} = \gamma^\mu. \tag{5.52}$$

In most relevant representations, in particular in the chiral one, one finds

$$\gamma_\dagger = \gamma_\dagger^{-1} = \gamma^0. \tag{5.53}$$

Therefore, one should modify hermitian conjugation for a spinor ψ and likewise for a spinor matrix X as

$$\bar{\psi} = \psi^\dagger\gamma_\dagger^{-1}, \qquad \bar{X} = \gamma_\dagger X \gamma_\dagger^{-1}. \tag{5.54}$$

The gamma matrices are self-adjoint under modified hermitian conjugation, $\bar{\gamma}^\mu = \gamma^\mu$.

5.4   Spin Statistics

So far we have only considered the Dirac equation. For quantisation, conserved charges and later for adding interactions we should construct a Lagrangian.

Lagrangian. It is straight-forward to guess

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi. \tag{5.55}$$

The variation w.r.t. $\psi^\dagger$ obviously yields the Dirac equation. Variation w.r.t. ψ gives the hermitian conjugate equation

$$-i\partial_\mu\bar{\psi}\gamma^\mu - m\bar{\psi} = (i\gamma^\mu\partial_\mu\psi - m\psi)^\dagger\gamma^0 = 0 \tag{5.56}$$

In fact, the Lagrangian is almost real

$$
\begin{aligned}
\mathcal{L}^\dagger &= \psi^\dagger(-i(\gamma^\mu)^\dagger\partial_\mu^\dagger - m)\gamma^0\psi = \bar{\psi}(-i\gamma^\mu\partial_\mu^\dagger - m)\psi \\
&= -i\partial_\mu(\bar{\psi}\gamma^\mu\psi) + \mathcal{L}.
\end{aligned} \tag{5.57}
$$

There is merely an imaginary topological term left,[13] the action is manifestly real.

Hamiltonian Formulation. To go to the Hamiltonian framework we compute the conjugate momenta $\pi = \partial\mathcal{L}/\partial\dot{\psi} = i\psi^\dagger$ and $\pi^\dagger = \partial\mathcal{L}/\partial\dot{\psi}^\dagger = 0$. It turns out that the conjugate momenta are proportional to the fields:[14]

- Dirac equation is a first-order differential equation.
- There are no independent momenta.
- Phase space equals position space.

Keep in mind that ψ and $\psi^\dagger$ are canonically conjugate fields.

Instead of computing the Hamiltonian, we can compute the energy-momentum tensor of which it is a component

$$
T^{\mu\nu} = i\bar{\psi}\gamma^\mu\partial^\nu\psi - \eta^{\mu\nu}\mathcal{L}. \tag{5.58}
$$

Unfortunately this tensor is not symmetric as it should for Lorentz invariance. Gladly, the anti-symmetric part can be written as (making use of the e.o.m.)

$$
T^{[\mu\nu]} = i\partial_\rho(\bar{\psi}\gamma^{[\rho}\gamma^\mu\gamma^{\nu]}\psi) = \partial_\rho K^{\rho\mu\nu}. \tag{5.59}
$$

The contribution from K is a boundary term for the integral defining the total momentum integral P^μ. We should thus subtract $\partial_\rho K^{\rho\mu\nu}$ from $T^{\mu\nu}$.

The Hamiltonian for the Dirac equation now reads

$$
H = -\int d^3x\, T^{00} = \int d^3x\, \bar{\psi}\big(-i\vec{\gamma}\cdot\vec{\partial} + m\big)\psi. \tag{5.60}
$$

Charge Conjugation. A conventional treatment and conventional quantisation of the above framework of the Dirac equation leads to several undesirable features.

For example, the problem manifests for charge conjugation. For every solution ψ of the Dirac equation, there is a charge conjugate solution $\psi_C = \gamma_C\psi^{\dagger T}$. Let us

---

[13]The topological term can be removed from the Lagrangian to obtain a manifestly real $\mathcal{L}' = \mathcal{L} - \frac{i}{2}\partial_\mu(\bar{\psi}\gamma^\mu\psi)$.

[14]Here, $\pi^\dagger$ is not the complex conjugate of π because $\mathcal{L}$ is not real. For the real $\mathcal{L}'$ we get instead $\pi = \frac{i}{2}\psi^\dagger$ and $\pi^\dagger = -\frac{i}{2}\psi$.

compute its energy

$$
\begin{aligned}
H[\psi_{\mathrm{C}}] &= \int d^3x \, \psi^{\mathrm{T}} (\gamma_{\mathrm{C}})^\dagger \gamma^0 \big(-i\vec{\gamma}\cdot\vec{\partial} + m\big)\gamma_{\mathrm{C}}\psi^{\dagger\mathrm{T}} \\
&= \int d^3x \, \psi^{\mathrm{T}}(\gamma_{\mathrm{C}})^\dagger \gamma^0 \gamma_{\mathrm{C}} \big(+i(\vec{\gamma})^*\cdot\vec{\partial} + m\big)\psi^{\dagger\mathrm{T}} \\
&= -\int d^3x \, \psi^{\mathrm{T}}\gamma^0 \big(+i(\vec{\gamma})^*\cdot\vec{\partial} + m\big)\psi^{\dagger\mathrm{T}} \\
&= -\int d^3x \, \psi^{\mathrm{T}} \big(+i(\vec{\gamma})^{\mathrm{T}}\cdot\vec{\partial} + m\big)\gamma^0 \psi^{\dagger\mathrm{T}} \\
&\stackrel{*}{=} -\int d^3x \, \psi^\dagger \gamma^0 \big(+i\vec{\gamma}\cdot\vec{\partial}^{\,\mathrm{T}} + m\big)\psi \\
&= -\int d^3x \, \bar{\psi}\big(-i\vec{\gamma}\cdot\vec{\partial} + m\big)\psi = -H[\psi] \quad\quad (5.61)
\end{aligned}
$$

The charge conjugate solution has opposite energy

$$
H[\psi_{\mathrm{C}}] = -H[\psi]. \quad\quad (5.62)
$$

Compare to the scalar field and $\phi_{\mathrm{C}} = \phi^*$. There we have

$$
H[\phi_{\mathrm{C}}] = +H[\phi]. \quad\quad (5.63)
$$

This is consistent with positivity of the energy. Naively, the field ψ could not possibly have positive definite energy.

Related issues arise for propagators and causality.

All above steps are elementary. Only one step $(*)$ can be changed: Transposition. We have

$$
\psi^{\mathrm{T}} X \psi^{\dagger\mathrm{T}} = \psi^a X_a{}^b \psi_b^\dagger = \psi_b^\dagger X_a{}^b \psi^a = \psi^\dagger X^{\mathrm{T}} \psi. \quad\quad (5.64)
$$

Instead of $\psi^a \psi_b^\dagger = \psi_b^\dagger \psi^a$ could use a different rule [15]

$$
\psi^a \psi_b^\dagger = -\psi_b^\dagger \psi^a. \quad\quad (5.65)
$$

This inserts a minus sign at $*$ and the energy of a solution and its charge conjugate are the same

$$
H[\psi_{\mathrm{C}}] = +H[\psi]. \quad\quad (5.66)
$$

This solves all the issues of the spinor field.

Spin-Statistics Theorem. The spin-statistics theorem states that consistent quantisation of fields with half-integer spin requires the use of anti-commutation relations

$$
\{\psi, \psi^\dagger\} \sim \hbar. \quad\quad (5.67)
$$

Such fields are called fermionic, they obey the Fermi-Dirac statistics. Multi-particle wave functions will be totally anti-symmetric.

---

[15]Eventually, quantisation will make ψ's become operators which do not commute either.

Conversely, fields with integer spin require commutation relations

$$[\phi, \phi^\dagger] \sim \hbar. \tag{5.68}$$

These fields are called bosonic, they obey the Bose-Einstein statistics. Multi-particle wave functions will be totally symmetric.

5.5 Grassmann Numbers

Quantisation should be viewed as a deformation of classical physics. Therefore, the anti-commutation relations of the quantum theory $\{\psi, \psi^\dagger\} \sim \hbar$ should be reflected by anti-commuting fields $\{\psi, \psi^\dagger\} = 0$ in the classical theory. More generally,

$$\psi^a \psi^b = -\psi^b \psi^a, \qquad \psi^a \psi_b^\dagger = -\psi_b^\dagger \psi^a, \qquad \psi_a^\dagger \psi_b^\dagger = -\psi_b^\dagger \psi_a^\dagger. \tag{5.69}$$

Besides these additional signs, the fields ψ will commute with numbers and scalar fields.

We therefore cannot use ordinary commuting numbers to represent the field ψ in the classical Lagrangian, we need something else.

Description. The required extension of the concept of numbers is called Grassmann numbers:

- Grassmann numbers form a non-commutative ring with $\mathbb{Z}_2$ grading.
- Grassmann numbers are $\mathbb{Z}_2$-graded, they can be even or odd: $|a| = 0, 1$.[16]
- Sums and products respect the even/odd grading

$$|a + b| = |a| = |b|, \qquad |ab| = |a| + |b|. \tag{5.70}$$

- The product is commutative unless both factors are odd in which case it is anti-commutative

$$ab = (-1)^{|a||b|} ba. \tag{5.71}$$

- Ordinary numbers are among the even Grassmann numbers.
- The field ψ takes values in odd Grassmann numbers.
- Real and complex Grassmann numbers can be defined. Grassmann numbers then form an algebra over the respective field.

A basis a^n of odd Grassmann numbers can be constructed out of a Clifford algebra $\{\gamma^j, \gamma^k\} = 2\delta^{jk}$

$$a^n = \frac{1}{\sqrt{2}} \left(\gamma^{2n} + i\gamma^{2n+1} \right). \tag{5.72}$$

In other words, Grassmann numbers can be represented in terms of (large) matrices. One should view the basis a^n to be sufficiently large or infinite.[17]

---

[16] Linear combinations of even and odd numbers could be defined, but usually they do not appear.

[17] There is no distinguished element such as i which extends the real numbers to complex numbers. We therefore do not have universal means to assign a value to a Grassmann variable. We will mainly use Grassmann variables to describe classical (fermionic) fields without assigning values.

Calculus. One can do calculus with Grassmann numbers much like ordinary numbers, but note:

- odd numbers square to zero: $(a^n)^2 = \frac{1}{2}\{a^n, a^n\} = 0$.
- the square root of zero is ill-defined.
- odd numbers have no inverse.
- some even numbers (e.g. products of two odd numbers) have no inverse.

A derivative for odd numbers can be defined as usual

$$\frac{\partial}{\partial a^m} a^n = \delta^n_m. \tag{5.73}$$

Note that derivatives are also odd objects

$$\{\partial/\partial a^m, \partial/\partial a^n\} = 0. \tag{5.74}$$

The above relation should be written as

$$\{\partial/\partial a^m, a^n\} = \delta^n_m. \tag{5.75}$$

We can also define the derivatives as elements of the same Clifford algebra

$$\frac{\partial}{\partial a^n} = \frac{1}{\sqrt{2}}\left(\gamma^{2n} - i\gamma^{2n-1}\right). \tag{5.76}$$

This leads to the same anti-commutation relations as above.

Complex Conjugation. A complex Grassmann number a can be written as a combination of the real Grassmann numbers $a_{\mathrm{r}}, a_{\mathrm{i}}$ as

$$a = a_{\mathrm{r}} + ia_{\mathrm{i}}. \tag{5.77}$$

Spinor fields are typically complex and we often need to complex conjugate them. Confusingly, there are two equivalent definitions of complex conjugation for Grassmann numbers.

One is reminiscent of complex conjugation

$$a^* = a_{\mathrm{r}} - ia_{\mathrm{i}}. \tag{5.78}$$

It obviously satisfies

$$(ab)^* = a^* b^*. \tag{5.79}$$

The other conjugation is reminiscent of hermitian adjoint. It satisfies

$$(ab)^\dagger = b^\dagger a^\dagger. \tag{5.80}$$

For ordinary numbers it would be the same as complex conjugation, but odd Grassmann numbers do not commute. The two definitions can be related as follows

$$a^\dagger = \begin{cases} a^* & \text{if } a \text{ is even,} \\ -ia^* & \text{if } a \text{ is odd.} \end{cases} \tag{5.81}$$

QM and QFT frequently use the adjoint operation, hence it is convenient to also use the adjoint for Grassmann numbers.

One should pay attention in defining real and odd Grassmann numbers. An odd number which satisfies $a^\dagger = a$ is not real. In particular, the even product of two such numbers is imaginary

$$(ab)^\dagger = b^\dagger a^\dagger = ba = -ab. \tag{5.82}$$

Instead real odd numbers are defined by

$$a^\dagger = -ia^* = -ia. \tag{5.83}$$

5.6 Quantisation

Poisson Brackets. We have also seen that ψ and $\psi^\dagger$ are canonically conjugate fields, there is no need to introduce additional conjugate momenta. The Poisson bracket for the spinor field should read[18]

$$\{F, G\} = i \int d^3x \left(\frac{\delta F}{\delta \psi^a(\vec{x})} \frac{\delta G}{\delta \psi_a^\dagger(\vec{x})} + \frac{\delta F}{\delta \psi_a^\dagger(\vec{x})} \frac{\delta G}{\delta \psi^a(\vec{x})} \right). \tag{5.84}$$

This expression can also be written as[19]

$$\{\psi^a(\vec{x}), \psi_b^\dagger(\vec{y})\} = \{\psi_b^\dagger(\vec{y}), \psi^a(\vec{x})\} = i\delta_b^a \delta^3(\vec{x} - \vec{y}). \tag{5.85}$$

Anti-Commutators. For quantisation, these Poisson brackets are replaced by an anti-commutator[20]

$$\{\psi^a(\vec{x}), \psi_b^\dagger(\vec{y})\} = \delta_b^a \delta^3(\vec{x} - \vec{y}). \tag{5.86}$$

By Fourier transformation to momentum space

$$\psi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 \, 2e(\vec{p})} \left(e^{-ip\cdot x} u_\alpha(\vec{p}) b_\alpha(\vec{p}) + e^{ip\cdot x} v_\alpha(\vec{p}) a_\alpha^\dagger(\vec{p}) \right) \tag{5.87}$$

we obtain anti-commutation relations for the Fourier modes

$$\{u_\gamma^a(\vec{p}) b_\gamma(\vec{p}), \bar{u}_{\gamma,b}(\vec{q}) b_\gamma^\dagger(\vec{q})\} = (p\cdot\gamma + m)^a{}_b \, 2e(\vec{p})(2\pi)^3 \delta^3(\vec{p} - \vec{q}).$$
$$\{\bar{v}_{\gamma,b}(\vec{p}) a_\gamma(\vec{p}), v_\gamma^a(\vec{q}) a^\dagger \gamma(\vec{q})\} = (p\cdot\gamma - m)^a{}_b \, 2e(\vec{p})(2\pi)^3 \delta^3(\vec{p} - \vec{q}). \tag{5.88}$$

It is convenient to split these relations into contributions from quantum operators and contributions from spinor solutions. We postulate simple anti-commutation relations for the creation and annihilation operators

$$\{a_\alpha(\vec{p}), a_\beta^\dagger(\vec{q})\} = \{b_\alpha(\vec{p}), b_\beta^\dagger(\vec{q})\} = \delta_{\alpha\beta} \, 2e(\vec{p})(2\pi)^3 \delta^3(\vec{p} - \vec{q}). \tag{5.89}$$

---

[18]The correct normalisation can be derived from $\dot{F} = -\{H, F\}$.

[19]Although the Poisson brackets are anti-symmetric in most cases, they are symmetric for two Grassmann odd elements.

[20]The sign is determined by the relation $[H, \psi^a(\vec{x})] = -i\dot{\psi}^a(\vec{x})$.

Together with the above anti-commutators, they imply the completeness relation for the basis of spinor solutions

$$u_\alpha(\vec{p})\bar{u}_\alpha(\vec{p}) = p{\cdot}\gamma + m,$$
$$v_\alpha(\vec{p})\bar{v}_\alpha(\vec{p}) = p{\cdot}\gamma - m. \tag{5.90}$$

In the Weyl representation these relations are reproduced by the particular choice for the spinors u, v.

$$u_\alpha(\vec{p}) = \begin{pmatrix} (p{\cdot}\sigma)^{1/2}\xi_\alpha \\ (p{\cdot}\bar{\sigma})^{1/2}\xi_\alpha \end{pmatrix}, \qquad v_\alpha(\vec{p}) = \begin{pmatrix} (p{\cdot}\sigma)^{1/2}\xi_\alpha \\ -(p{\cdot}\bar{\sigma})^{1/2}\xi_\alpha \end{pmatrix}, \tag{5.91}$$

where ξ_α is an orthonormal basis of 2-spinors.

Dirac Sea. The Pauli exclusion principle for fermions states that each state can be occupied only once. It follows from the (not explicitly written) anti-commutators

$$\{a_\alpha^\dagger(\vec{p}), a_\beta^\dagger(\vec{q})\} = \{b_\alpha^\dagger(\vec{p}), b_\beta^\dagger(\vec{q})\} = 0 \tag{5.92}$$

that

$$(a_\alpha^\dagger(\vec{p}))^2 = (b_\alpha^\dagger(\vec{p}))^2 = 0. \tag{5.93}$$

Dirac used the exclusion principle to make useful proposals concerning negative-energy states in relativistic quantum mechanics and the prediction of anti-particles.

The Dirac equation has positive and negative solutions. Furthermore the solutions carry an (electrical) charge. Dirac proposed that all negative-energy states are already occupied in the vacuum and cannot be excited further. This picture is called the Dirac sea, and it explained how to avoid negative-energy solutions.

Continuing this thought, there is now the option to remove an excitation from one of the occupied states. This hole state would not only have positive energy, but also carry charges exactly opposite to the ones of the regular positive-energy solutions. In this way he predicted the existence of positrons as the anti-particles of electrons. The prediction was soon thereafter confirmed in experiment.

Our view of QFT today is different, so let us compare:

- Positive-energy solutions of ψ are associated to $a^\dagger$.
- Negative-energy solutions of ψ are associated to b.
- Let us define $c^\dagger = b$, it is our choice.
- The vacuum is annihilated by $c^\dagger$. All c-states are occupied.
- A hole in the Dirac sea $c = b^\dagger$ creates an anti-particle.

QFT for Dirac particles works as predicted, but:

- There is no need for a Dirac sea.
- Negative-energy solutions are defined as annihilation operators, not as creation operators with occupied states.
- Dirac's argument relies on the exclusion principle, it works for fermions only. QFT can also deal with bosons.

- The Dirac equation has real solutions (see later) just as well as the Klein–Gordon equation has complex solutions.
  The existence of anti-particles is unrelated to spin and the Dirac equation. It is a consequence of CPT.

Correlators and Propagators. We now have all we need to compute correlators of the free quantum fields. There are two non-vanishing correlators of two fields

$$\Delta^{\mathrm{D}a}_{+b}(x-y) = \langle 0|\psi^a(x)\bar{\psi}_b(y)|0\rangle,$$
$$\Delta^{\mathrm{D}a}_{-b}(x-y) = \langle 0|\bar{\psi}_b(y)\psi^a(x)|0\rangle. \tag{5.94}$$

They can be expressed in terms of the correlator of two scalars Δ_+ with an additional operator

$$\Delta^{\mathrm{D}a}_{+b}(x) = \int \frac{d^3\vec{p}\, e^{-ip\cdot x}}{(2\pi)^3\, 2e(\vec{p})}\, (p\cdot\gamma + m)^a_{b} = (i\gamma\cdot\partial + m)^a_{b}\Delta_+(x),$$
$$\Delta^{\mathrm{D}a}_{-b}(-x) = \int \frac{d^3\vec{p}\, e^{-ip\cdot x}}{(2\pi)^3\, 2e(\vec{p})}\, (p\cdot\gamma - m)^a_{b} = (i\gamma\cdot\partial - m)^a_{b}\Delta_+(x). \tag{5.95}$$

When acting with the Dirac equation on the correlator, it combines with the operator to give the Klein–Gordon equation acting on Δ_+, e.g.

$$\big(i(\partial/\partial x^\mu)\gamma^\mu - m\big)^a_{b}\Delta^{\mathrm{D}b}_{+c}(x)$$
$$= (i\gamma\cdot\partial - m)^a_{b}(i\gamma\cdot\partial + m)^b_{c}\Delta_+(x)$$
$$= \delta^a_c\big((\partial/\partial x)^2 - m^2\big)\Delta_+(x) = 0. \tag{5.96}$$

Likewise, the unequal time anti-commutator

$$\{\psi^a(x), \bar{\psi}_b(y)\} = \Delta^{\mathrm{D}a}_{b}(y-x) \tag{5.97}$$

can be written in terms of the one for the scalar field

$$\Delta^{\mathrm{D}}(x) = (i\gamma\cdot\partial - m)\Delta(x). \tag{5.98}$$

As such it satisfies the Dirac equation and vanishes for space-like separations[21]

$$\{\psi^a(x), \bar{\psi}_b(y)\} = 0 \quad \text{for } (x-y)^2 > 0. \tag{5.99}$$

For the Dirac equation with a source, the same methods we introduced earlier for the scalar field apply. The propagator is a spinor matrix and defined via the equations

$$(-i\gamma\cdot\partial + m)^a_{b}G^{\mathrm{D}b}_{c}(x) = \delta^a_c\delta^4(x),$$
$$G^{\mathrm{D}a}_{b}(x)(-i\gamma\cdot\partial^\dagger + m)^b_{c} = \delta^a_c\delta^4(x), \tag{5.100}$$

---

[21]The fact that an anti-commutator vanishes is not in contradiction with causality. Typically we can observe only fermion bilinears which are bosonic and do commute.

supplemented by suitable boundary conditions. By the same reasons as above, we can express the Dirac propagator through the scalar propagator

$$G^{\mathrm{D}a}{}_b(x) = (i\gamma{\cdot}\partial + m)^a{}_b G(x). \tag{5.101}$$

Obviously, one has the same relations as before, e.g. for the retarded propagator

$$G^{\mathrm{D}}_{\mathrm{R}}(x) = i\theta(t)\Delta^{\mathrm{D}}(x). \tag{5.102}$$

There are some other useful relationships between correlators and propagators in momentum space which are worth emphasising because they hold generally.

First of all, by construction the propagator is the inverse of the kinetic term in the action[22]

$$G^{\mathrm{D}}(p) = (-\gamma{\cdot}p + m)^{-1} = \frac{\gamma{\cdot}p + m}{p^2 + m^2} \tag{5.103}$$

The corresponding correlators and unequal time commutators take the form

$$\Delta^{\mathrm{D}}_\pm(p) = \pm 2\pi\delta(p^2 + m^2)\theta(\pm p_0)(p{\cdot}\gamma + m),$$
$$\Delta^{\mathrm{D}}(p) = 2\pi\delta(p^2 + m^2)\,\mathrm{sign}(p_0)(p{\cdot}\gamma + m). \tag{5.104}$$

These reflect precisely the residues times a delta function localised at the position of the pole along with some restriction to positive or negative energies.

The construction of the propagator and its relationship to correlators and commutators can be used as a shortcut in deriving the latter. Large parts of the canonical quantisation procedure can thus be avoided in practice.

5.7 Complex Field

The Dirac spinor is complex and the Lagrangian has the obvious U(1) global symmetry

$$\psi'(x) = e^{i\alpha}\psi, \qquad \bar\psi'(x) = e^{-i\alpha}\bar\psi. \tag{5.105}$$

The symmetry has a corresponding conserved Noether current[23]

$$J^\mu = \frac{\delta\psi^a}{\delta\alpha}\frac{\delta\mathcal{L}}{\delta\partial_\mu\psi^a} = -\bar\psi\gamma^\mu\psi. \tag{5.106}$$

The time component of the current was used earlier to define a positive definite probability density, $-J^0 = \psi^\dagger\psi$. However, if one follows the spin-statistics theorem and let ψ be Grassmann odd, the density is not positive. In particular, it changes sign for the charge conjugate solution

$$J^\mu_{\mathrm{C}} = -\bar\psi_{\mathrm{C}}\gamma^\mu\psi_{\mathrm{C}} = \bar\psi\gamma^\mu\psi = -J^\mu. \tag{5.107}$$

---

[22]The poles should be shifted away from the real axis to accommodate for the desired boundary conditions.

[23]Note the order of terms.

Nevertheless, the current is conserved and defines a conserved Noether charge

$$Q = \int d^3\vec{x}\, J^0 = -\int d^3\vec{x}\, \psi^\dagger \psi. \qquad (5.108)$$

It leads to the usual charge assignments for the complex field

$$[Q, \psi(x)] = \psi(x), \qquad [Q, \bar{\psi}(x)] = -\bar{\psi}(x). \qquad (5.109)$$

5.8 Real Field

The Dirac field has four independent particle modes, $a_\alpha^\dagger(\vec{p})$ and $b_\alpha^\dagger(\vec{p})$, for each three-momentum. From the classification of Poincaré UIR's we know that the irreducible representation for spin $j = \frac{1}{2}$ has only two spin orientations for each three-momentum.

Reality Condition. This discrepancy is associated to the existence of charge conjugate solutions. We can remove the additional solutions by imposing a reality condition on ψ, namely[24]

$$\psi_{\rm C} = \psi. \qquad (5.110)$$

A spinor which satisfies this condition is called a Majorana spinor. There exist representations of the Clifford algebra where all γ^μ are purely imaginary. In this basis the Dirac equation is real, and it makes sense to restrict ψ to real (Grassmann odd) numbers.

For the momentum modes we can use the identity

$$u_\alpha(\vec{p}) = \gamma_{\rm C} v_{\alpha'}^*(\vec{p}), \qquad (5.111)$$

to show that the identification $\psi_{\rm C} = \psi$ implies

$$a_\alpha(\vec{p}) = b_{\alpha'}(\vec{p}) \qquad (5.112)$$

The identification may involve some translation between the bases a_α and $b_{\alpha'}$. It reduces the modes of the Dirac field by a factor of two.

2-Spinors. Let us consider a real spinor $\psi = (\psi_{\rm L}, \psi_{\rm R})$ in the Weyl representation. The reality condition implies

$$\psi_{\rm L} = -i\sigma^2 \psi_{\rm R}^{\dagger\rm T} = \frac{1}{\sqrt{2}}\chi. \qquad (5.113)$$

This allows to write the Lagrangian in terms of the 2-spinor field χ as[25]

$$\mathcal{L} = \chi^\dagger i\bar{\sigma}\cdot\partial\chi + \tfrac{i}{2}m\chi^{\rm T}\sigma^2\chi - \tfrac{i}{2}m\chi^\dagger\sigma^2\chi^{\dagger\rm T}. \qquad (5.114)$$

The Lagrangian for the Dirac field can be written as two identical copies of this.[26]

---

[24]One could also use any other complex phase $e^{i\alpha}$ between $\psi_{\rm C}$ and ψ.

[25]The two mass terms are anti-symmetric in χ, which requires the classical field χ to be an odd Grassmann number.

[26]The U(1) global symmetry of the Dirac equation is recovered as a SO(2) rotation symmetry of the two fields χ.

Parity. Note that parity interchanges ψ_L and ψ_R. The reality condition relates the two, hence

$$\chi'(t, -\vec{x}) = -i\sigma^2 \chi^{\dagger\mathrm{T}}(t, \vec{x}). \tag{5.115}$$

As this transformation also sends the field χ to its complex conjugate $\chi^{\dagger\mathrm{T}}$, it is usually viewed as the combination CP of charge conjugation C and parity P

$$CP\chi(t, \vec{x})(CP)^{-1} = -i\sigma^2 \chi^{\dagger\mathrm{T}}(t, -\vec{x}). \tag{5.116}$$

In that sense there cannot be individual C and P transformations and only CP can be a symmetry.

An alternative point of view is that C was used to define the reality condition. Hence C is preserved by construction, and the parity operation P is well-defined on its own.

Technically, both points of view have the same content: They merely use the same words to refer to different operators. They are related by identifying $C' = 1$ and $P' = CP$, where the primed operations refer to the latter approach.

5.9 Massless Real Field

So far we have assumed a non-zero mass m. Let us now consider the massless case which has some special features. We will assume a real (Majorana) field.

First, let us compare to the UIR's of the Poincaré group: There are two particles $a_\alpha^\dagger(\vec{p})$ for each momentum. Conversely, a UIR with fixed helicity has merely one state for each momentum. The two particles correspond to UIR's with helicity $h = \pm\frac{1}{2}$. In fact, helicity states must always come in pairs in QFT. One cannot construct a real Lagrangian which describes just one helicity.

Interestingly, the splitting of representations leads to an enhancement of symmetry. For $m = 0$, the Lagrangian in terms of 2-spinors reads

$$\mathcal{L} = \chi^\dagger i\bar{\sigma}\cdot\partial\chi. \tag{5.117}$$

Quite obviously, this Lagrangian has a global U(1) symmetry

$$\chi' = e^{i\alpha}\chi. \tag{5.118}$$

It is called chiral symmetry. The associated Noether current reads

$$J^\mu = -\chi^\dagger \bar{\sigma}^\mu \chi. \tag{5.119}$$

At the level of 4-spinors chiral symmetry is represented by the transformation

$$\psi' = \exp(-i\alpha\gamma^5)\psi. \tag{5.120}$$

The adjoint spinor transforms with the same factor

$$\bar{\psi}' = \bar{\psi}\exp(-i\alpha\gamma^5). \tag{5.121}$$

This is transformed to the inverse factor by a single gamma matrix

$$\exp(-i\alpha\gamma^5)\gamma^\mu = \gamma^\mu \exp(i\alpha\gamma^5). \qquad (5.122)$$

The massless Lagrangian is therefore invariant under chiral transformations. Here the conserved current is the so-called axial vector current

$$J^\mu = -\bar{\psi}\gamma^5\gamma^\mu\psi. \qquad (5.123)$$

5.10   Chiral Transformations and Masses

Let us consider the above chiral transformations in the presence of mass

$$\begin{aligned}\mathcal{L}' &= \bar{\psi}\big(i\gamma\!\cdot\!\partial - m\exp(2i\alpha\gamma^5)\big)\psi \\ &= \bar{\psi}\big(i\gamma\!\cdot\!\partial - m\cos(2\alpha) - m\sin(2\alpha)i\gamma^5\big)\psi. \end{aligned} \qquad (5.124)$$

On the one hand, it shows that masses break chiral symmetry. On the other hand, we have learned that there are two types of mass terms

$$\bar{\psi}\psi \quad \text{and} \quad i\bar{\psi}\gamma^5\psi. \qquad (5.125)$$

They are both equivalent under a chiral transformation. When they appear simultaneously, the physical squared mass is the sum of the squares of the two coefficients.

For a real field, the transformed Lagrangian in terms of 2-spinors reads

$$\mathcal{L}' = \chi^\dagger i\bar{\sigma}\!\cdot\!\partial\chi + \tfrac{i}{2}me^{2i\alpha}\chi^{\mathrm{T}}\sigma^2\chi - \tfrac{i}{2}me^{-2i\alpha}\chi^\dagger\sigma^2\chi^{\dagger\mathrm{T}}. \qquad (5.126)$$

We could thus also introduce a complex m such that the physically relevant squared mass is just $m\bar{m} = |m|^2$.

6 Free Vector Field

Next we want to find a formulation for vector fields. This includes the important case of the electromagnetic field with its photon excitations as massless relativistic particles of helicity 1. This field will be the foundation for a QFT treatment of electrodynamics called quantum electrodynamics (QED). Here we will encounter a new type of symmetry which will turn out to be extremely powerful but at the price of new complications.

6.1 Classical Electrodynamics

We start by recalling electrodynamics which is the first classical field theory most of us have encountered in theoretical physics.

Maxwell Equations. The electromagnetic field consists of the electric field $\vec{E}(t, \vec{x})$ and the magnetic field $\vec{B}(t, \vec{x})$. These fields satisfy the four Maxwell equations (with $\varepsilon_0 = \mu_0 = c = 1$)

$$
\begin{aligned}
0 &= \operatorname{div} \vec{B} := \vec{\partial}{\cdot}\vec{B} = \partial_k B_k, \\
0 &= \operatorname{rot} \vec{E} + \dot{\vec{B}} := \vec{\partial} \times \vec{E} + \dot{\vec{B}} = \varepsilon_{ijk}\partial_j E_k + \dot{B}_i, \\
\rho &= \operatorname{div} \vec{E} = \vec{\partial}{\cdot}\vec{E} = \partial_k E_k, \\
\vec{\jmath} &= \operatorname{rot} \vec{B} - \dot{\vec{E}} = \vec{\partial} \times \vec{B} - \dot{\vec{E}} = \varepsilon_{ijk}\partial_j B_k - \dot{E}_i.
\end{aligned}
\tag{6.1}
$$

The fields ρ and $\vec{\jmath}$ are the electrical charge and current densities.

The solutions to the Maxwell equations without sources are waves propagating with the speed of light. The Maxwell equations were the first relativistic wave equations that were found. Eventually their consideration led to the discovery of special relativity.

Relativistic Formulation. Lorentz invariance of the Maxwell equations is not evident in their usual form. Let us transform them to a relativistic form.

The first step consists in converting B_i to an anti-symmetric tensor of rank 2

$$
B_i = -\tfrac{1}{2}\varepsilon_{ijk}F_{jk}, \qquad F = \begin{pmatrix} 0 & -B_z & +B_y \\ +B_z & 0 & -B_x \\ -B_y & +B_x & 0 \end{pmatrix}.
\tag{6.2}
$$

Then the Maxwell equations read

$$
\begin{aligned}
0 &= -\varepsilon^{ijk}\partial_k F_{ij}, & \rho &= \partial_k E_k, \\
0 &= \varepsilon^{ijk}(2\partial_j E_k - \dot{F}_{jk}), & \vec{\jmath} &= \partial_j F_{ji} - \dot{E}_i.
\end{aligned}
\tag{6.3}
$$

These equations are the $1 + 3$ components of two 4-vectors which can be seen by setting

$$E_k = F_{0k} = -F_{k0}, \qquad J^\mu = (\rho, \vec{\jmath}). \tag{6.4}$$

Now the Maxwell equations simply read

$$\varepsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = 0, \qquad \partial_\nu F^{\nu\mu} = J^\mu. \tag{6.5}$$

Electromagnetic Potential. For QFT purposes we need to write a Lagrangian from which the Maxwell equations follow. This is however not possible using $F_{\mu\nu}$ as the fundamental degrees of freedom. A Lagrangian can be constructed by the help of the electromagnetic vector potential A_μ. This is not just a technical tool, but it will be necessary to couple the field to charged matter. This fact can be observed in the Aharonov–Bohm effect, where a quantum particle feels the presence of a non-trivial electromagnetic potential A, although it is confined to a region of spacetime where the field strength vanishes $F = 0$.

The first (homogeneous) equation is an integrability condition for the field $F_{\mu\nu}$. It implies that it can be integrated consistently to an electromagnetic potential A_μ

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{6.6}$$

With this parametrisation of F the homogeneous equation is automatically satisfied.

The electromagnetic potential is not uniquely defined by the electromagnetic fields F. For any solution A, we can add the derivative of a scalar field

$$A'_\mu(x) = A_\mu(x) + \partial_\mu \alpha(x). \tag{6.7}$$

The extra term cancels out when anti-symmetrising the two indices in $F_{\mu\nu}$ and hence

$$F'_{\mu\nu}(x) = F_{\mu\nu}(x). \tag{6.8}$$

This freedom in defining A_μ is called a gauge symmetry or gauge redundancy. It is called a local symmetry because the transformation can be chosen independently for every point of spacetime. Gauge symmetry will turn out very important in quantising the vector field.

Lagrangian. A Lagrangian for the electromagnetic fields can now be formulated in terms of the potential A_μ

$$\mathcal{L} = -\tfrac{1}{4} F^{\mu\nu}[A] F_{\mu\nu}[A] = \tfrac{1}{2} \vec{E}[A]^2 - \tfrac{1}{2} \vec{B}[A]^2. \tag{6.9}$$

Here and in the following, $F_{\mu\nu}[A]$ is not considered a fundamental field, but merely represents the combination

$$F_{\mu\nu}[A] = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{6.10}$$

The equation of motion yields the second (inhomogeneous) Maxwell equation (here with a trivial source term)

$$\partial_\nu F^{\nu\mu} = 0. \tag{6.11}$$

The first (homogeneous) Maxwell equation is already implied by the definition of F in terms of A.

Due to Poincaré symmetry we can also derive an energy momentum tensor $T_{\mu\nu}$. It takes the form[1]

$$T_{\mu\nu} = -F_\mu{}^\rho F_{\nu\rho} + \tfrac{1}{4}\eta_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}. \tag{6.12}$$

6.2 Gauge Fixing

Hamiltonian Framework. Towards quantisation we should proceed to the Hamiltonian framework. The canonical momentum Π conjugate to the field A reads

$$\Pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}^\mu} = F_{0\mu}. \tag{6.13}$$

Here a complication arises because the component Π_0 is strictly zero and the field A_0 has no conjugate momentum. The non-zero components are the electrical field $F_{0k} = E_k$. Moreover the equations of motion imply $\partial_k \Pi_k = 0$ which is an equation without time derivative.

The missing of the momentum Π_0 and the spatial differential equation for Π_k are two constraints which the momenta will have to satisfy, even in the initial condition. These are so-called constraints. For the massless vector field they are related to gauge redundancy of A. Although A_μ has four components, one of them can be chosen arbitrarily using gauge symmetry. Effectively A_μ has only three physically relevant components, which is matched by only three conjugate momenta.

Coulomb Gauge. A simple ansatz to resolve the problem of $\Pi_0 = 0$ is to demand that

$$A_0 = 0. \tag{6.14}$$

This can always be achieved by a suitable gauge transformation.

This choice does not completely eliminate all gauge freedom for A_k, a time-independent gauge redundancy $\alpha(\vec{x})$ remains. It can be eliminated by the demanding

$$\partial_k A_k = 0 \tag{6.15}$$

which is called the Coulomb gauge (fixing).

Now $\Pi_k = F_{0k} = E_k = \dot{A}_k$ and for the Hamiltonian we obtain

$$H = \int d^3x \, \tfrac{1}{2}(\vec{E}^2 + \vec{B}^2), \tag{6.16}$$

---

[1]The naive derivation from the Lagrangian yields $T_{\mu\nu} = -F_{\mu\rho}\partial_\nu A^\rho - \tfrac{1}{4}\mathcal{L}$ which is neither symmetric nor gauge invariant. It is repaired by adding the term $\partial^\rho(F_{\mu\rho}A_\nu)$.

which indeed represents the energy of the electromagnetic field.

With the Coulomb gauge, we can now quantise the electromagnetic field. The gauge is however not always convenient, since it specialises the time direction and therefore breaks relativistic invariance. For instance, it leads to instantaneous contributions to field correlators, which may feel odd. In physical observables, eventually such instantaneous or causality-violating contributions will always cancel.

Lorenz Gauges. A more general class gauge fixings are the Lorenz gauges

$$\partial^\mu A_\mu = 0. \tag{6.17}$$

Again, they do not completely fix the gauge freedom since any gauge transformation with $\partial^2 \alpha = 0$ will preserve the Lorenz gauge condition. For example, one may furthermore demand $A_0 = 0$ to recover the Coulomb gauge.

It is convenient to implement the Lorenz gauge by adding a gauge fixing term $\mathcal{L}_{\text{gf}} = -\frac{1}{2}\xi(\partial{\cdot}A)^2$ to the Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{ED}} + \mathcal{L}_{\text{gf}} \simeq -\tfrac{1}{2}\partial^\mu A^\nu \, \partial_\mu A_\nu + \tfrac{1}{2}(1-\xi)\partial^\mu A_\mu \partial^\nu A_\nu. \tag{6.18}$$

The equations of motion now read

$$\partial^2 A_\mu - (1-\xi)\partial_\mu \partial^\nu A_\nu = 0 \tag{6.19}$$

Let us show that the equations require $\partial^2 A_\mu = 0$ and $\partial \cdot A = 0$. We solve the equation in momentum space

$$p^2 A_\mu - (1-\xi)p_\mu(p{\cdot}A) = 0 \tag{6.20}$$

We first multiply the equation by p^μ to get

$$\xi p^2(p{\cdot}A) = 0. \tag{6.21}$$

Unless $\xi = 0$, this equation implies that $p^2 = 0$ or $p{\cdot}A = 0$. Using this result in the original equation shows that both $p^2 = 0$ and $p{\cdot}A = 0$ must hold unless $\xi = 1$. [2] For $\xi = 1$, the equation only requires $p^2 = 0$.

The canonical momenta now read

$$\Pi_\mu = \dot{A}_\mu + \delta_\mu^0(1-\xi)(\partial^\nu A_\nu), \tag{6.22}$$

which can be solved for all $\dot{A}_\mu$ unless $\xi = 0$. We can now define canonical Poisson brackets

$$\{A_\mu(t,\vec{x}), \Pi^\nu(t,\vec{y})\} = \delta_\mu^\nu \delta^3(\vec{x} - \vec{y}). \tag{6.23}$$

---

[2]This statement might not be accurate in a distributional sense! For example, the function $A_\mu = x_\mu$ satisfies the equation of motion, but yields $\partial{\cdot}A = -4 \neq 0$. In momentum space this function is a distribution.

There appears to be a catch: The gauge fixing condition $\partial \cdot A = 0$ constrains the conjugate momentum Π_0

$$\Pi_0 = \partial_k A_k - \xi(\partial^\nu A_\nu). \tag{6.24}$$

to be a function of the A_k alone. Substituting this into the Poisson brackets yields the identity

$$\{A_0, \Pi_0\} = \{A_0, \partial_k A_k\} - \xi\{A_0, \partial^\nu A_\nu\} = -\xi\{A_0, \partial^\nu A_\nu\}, \tag{6.25}$$

where we used that the A's and their spatial derivatives commute. Here the term on the left hand side is non-trivial whereas the term on the right hand side should be trivial due to $\partial \cdot A = 0$.[3]

Feynman Gauge. To avoid these problems and also to simplify the subsequent analysis, we shall set $\xi = 1$. This so-called Feynman gauge[4] has a simple Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu A^\nu \, \partial_\mu A_\nu \tag{6.26}$$

with simple equations of motion

$$\partial^2 A_\mu = 0. \tag{6.27}$$

Effectively, it describes 4 massless scalar fields A_μ with the peculiarity that the sign of the kinetic term for A_0 is wrong. With the canonical momenta $\Pi_\mu = \dot{A}_\mu$ the Poisson brackets read

$$\{A_\mu(\vec{x}), \Pi_\nu(\vec{y})\} = \eta_{\mu\nu}\delta^3(\vec{x} - \vec{y}). \tag{6.28}$$

where again the relation for A_0 has the opposite sign. Likewise all correlation functions and propagators equal their scalar counterparts times $\eta_{\mu\nu}$.

As such, the model described by the above simple Lagrangian is not electrodynamics. Only when taking into account the constraint $\partial \cdot A = 0$ it becomes electrodynamics. Moreover, the constraint will be crucial in making the QFT model physically meaningful. Nevertheless we have to be careful in implementing the constraint since it is inconsistent with the Poisson brackets.

Light Cone Gauge. The above Lorenz gauges do not eliminate all unphysical degrees of freedom, which introduce some complications later. There are other useful gauges which avoid these problems, but trade them in for others. A prominent example is the light cone gauge which eliminates a light-like component $A_- = A_0 - A_3 = 0$ of the gauge potential A_μ. The equations of motion then allow to solve for a non-collinear like-like component of $A_+ = A_0 + A_3$. The remaining two degrees of freedom of A_μ then represent the two helicity modes of the electromagnetic field. Let us nevertheless continue in the Feynman gauge.

---

[3]However, note that $\partial \cdot A = 0$ need not hold in a strict sense, i.e. the above Poisson brackets are self-consistent and can be quantised.

[4]Actually, it is a gauge fixing term rather than a gauge.

6.3 Particle States

Next we quantise the model and discuss its particle states. The construction of Fock space is the same as for a set of four massless scalar fields, but we need to implement the gauge-fixing constraint.

Quantisation. We quantise the vector field $A_\mu(x)$ in Feynman gauge analogously to four independent scalar fields where merely one of the kinetic term has the opposite sign. This leads to the equal-time commutation relations

$$[A_\mu(t, \vec{x}), \dot{A}_\nu(t, \vec{y})] = i\eta_{\mu\nu}\delta^3(\vec{x} - \vec{y}). \tag{6.29}$$

We then solve the equation of motion $\partial^2 A_\mu = 0$ in momentum space

$$A_\mu(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 \, 2e(\vec{p})} \left(e^{-ip\cdot x} a_\mu(\vec{p}) + e^{ip\cdot x} a_\mu^\dagger(\vec{p}) \right) \tag{6.30}$$

and translate the above field commutators to commutators for creation and annihilation operators

$$[a_\mu(\vec{p}), a_\nu^\dagger(\vec{q})] = \eta_{\mu\nu} \, 2e(\vec{p}) \, (2\pi)^3 \, \delta^3(\vec{p} - \vec{q}). \tag{6.31}$$

Fock Space. We define the vacuum state $|0\rangle$ to be annihilated by all $a_\mu(\vec{p})$

$$a_\mu(\vec{p})|0\rangle = 0. \tag{6.32}$$

As before, multi-particle states are constructed by acting with the creation operators $a_\mu^\dagger(\vec{p})$ on the vacuum $|0\rangle$.

There are two problems with this naive Fock space. The first is that there ought to be only two states (with helicity $h = \pm 1$) for each momentum. Here we have introduced four states. The other problem is that one of these states has a negative norm: To see this we prepare a wave packet for $a_0^\dagger$

$$|f\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3 \, 2e(\vec{p})} \, f(\vec{p}) a_0^\dagger(\vec{p})|0\rangle. \tag{6.33}$$

The norm of this state is negative definite

$$\langle f|f\rangle = -\int \frac{d^3\vec{p}}{(2\pi)^3 \, 2e(\vec{p})} \, |f(\vec{p})|^2 < 0. \tag{6.34}$$

A negative-norm state violates the probabilistic interpretation of QFT, hence it must be avoided at all means.

Physical States. The above problems are eventually resolved by implementing the gauge-fixing constraint $\partial\cdot A = 0$ which we have not yet considered. This is not straight-forward:

6.6

- The commutation relations prevent us from implementing it at an operatorial level.
- We cannot implement it directly on states: E.g. requiring the vacuum $|0\rangle$ to be physical means setting $p{\cdot}a^\dagger(\vec{p})|0\rangle = 0$ which is inconsistent with the commutation relations.
- The weakest implementation is to demand that the expectation value of $\partial{\cdot}A$ vanishes for all physical states. This is the Gupta–Bleuler formalism.

For two physical states $|\Psi\rangle, |\Phi\rangle$ we thus demand

$$\langle\Phi|\partial{\cdot}A|\Psi\rangle = 0. \tag{6.35}$$

This is achieved by demanding

$$p{\cdot}a(\vec{p})|\Psi\rangle = 0 \tag{6.36}$$

for any physical state. An adjoint physical state then obeys $\langle\Phi|p{\cdot}a^\dagger(\vec{p}) = 0$.

- Both conditions together ensure that $\langle\Phi|\partial{\cdot}A|\Psi\rangle = 0$.
- Moreover, the vacuum is physical by construction.

We conclude that Fock space is too large. The space of physical states $|\Psi\rangle$ is a subspace of Fock space such that for all $\vec{p}$

$$p{\cdot}a(\vec{p})|\Psi\rangle = 0. \tag{6.37}$$

Nevertheless we cannot completely abandon the larger Fock space in favour of the smaller space of physical states. For instance, the action of $A_\mu(x)$ cannot be confined to the physical subspace since it does not commute with the operator $p{\cdot}a(\vec{p})$.

Evidently, the negative-norm state $|f\rangle$ discussed above is not physical since

$$p{\cdot}a(\vec{p})|f\rangle = f(\vec{p})e(\vec{p})|0\rangle. \tag{6.38}$$

It is an element of Fock space, but not of its physical subspace.

Basis of Polarisation Vectors.   To investigate the space of physical states further, we introduce a convenient basis for polarisation vectors $\epsilon_\mu^{(\alpha)}(\vec{p})$ of the vector field $a_\mu(\vec{p})$ and $a_\mu^\dagger(\vec{p})$ on the light cone $p^2 = 0$.[5]

We denote the four polarisations α by G for gauge, L for longitudinal and $1, 2$ for the two transverse directions.

- We first define $\epsilon^{(\mathrm{G})}$ as a light-like vector in the direction of p, e.g. $\epsilon^{(\mathrm{G})} = p$.
- We construct another light-like vector $\epsilon^{(\mathrm{L})}$ which has unit scalar product with $\epsilon^{(\mathrm{G})}$, i.e. $\epsilon^{(\mathrm{L})}{\cdot}\epsilon^{(\mathrm{G})} = 1$.
- We then construct two orthonormal space-like vectors $\epsilon^{(1,2)}$ which are also orthogonal to $\epsilon^{(\mathrm{G})}$ and $\epsilon^{(\mathrm{L})}$.

---

[5]The polarisation vectors are similar to the spinors $u(\vec{p})$ and $v(\vec{p})$ for the Dirac equation.

$$(6.39)$$

For example, suppose the light-like momentum is given by

$$p_\mu = (e, 0, 0, e). \tag{6.40}$$

Then we can define the following 4 vectors [6]

$$\begin{aligned}
\epsilon_\mu^{(\mathrm{G})}(\vec{p}) &= (e, 0, 0, e), \\
\epsilon_\mu^{(\mathrm{L})}(\vec{p}) &= (-1/2e, 0, 0, 1/2e), \\
\epsilon_\mu^{(1)}(\vec{p}) &= (0, 1, 0, 0), \\
\epsilon_\mu^{(2)}(\vec{p}) &= (0, 0, 1, 0).
\end{aligned} \tag{6.41}$$

These four polarisations define a complete basis for vector space. We can thus decompose the creation and annihilation operators as follows

$$a_\mu^{(\dagger)} = \epsilon_\mu^{(\mathrm{G})} a_{(\mathrm{L})}^{(\dagger)} + \epsilon_\mu^{(\mathrm{L})} a_{(\mathrm{G})}^{(\dagger)} + \epsilon_\mu^{(1)} a_{(1)}^{(\dagger)} + \epsilon_\mu^{(2)} a_{(2)}^{(\dagger)}. \tag{6.42}$$

Likewise we can write the commutation relations[7]

$$\begin{aligned}
[a_{(\mathrm{L})}(\vec{p}), a_{(\mathrm{G})}^\dagger(\vec{q})] &= [a_{(1)}(\vec{p}), a_{(1)}^\dagger(\vec{q})] = \\
[a_{(\mathrm{G})}(\vec{p}), a_{(\mathrm{L})}^\dagger(\vec{q})] &= [a_{(2)}(\vec{p}), a_{(2)}^\dagger(\vec{q})] = 2e(\vec{p})\,(2\pi)^3 \delta^3(\vec{p} - \vec{q}).
\end{aligned} \tag{6.43}$$

By construction we know that

$$p{\cdot}a(\vec{p}) = \epsilon^{(\mathrm{G})}{\cdot}a(\vec{p}) = a_{(\mathrm{G})}(\vec{p}). \tag{6.44}$$

hence the physical state condition in this basis reads

$$a_{(\mathrm{G})}(\vec{p})|\Psi\rangle = 0. \tag{6.45}$$

The physical state condition together with the commutation relations implies that a physical state cannot have any longitudinal excitations $a_{(\mathrm{L})}^\dagger(\vec{p})$. It must be of the form[8]

$$|\Psi\rangle = a_{(\mathrm{G})}^\dagger \cdots a_{(\mathrm{G})}^\dagger a_{(1,2)}^\dagger \cdots a_{(1,2)}^\dagger |0\rangle. \tag{6.46}$$

Since negative norm states can originate exclusively from the commutators $[a_{(\mathrm{L})}, a_{(\mathrm{G})}^\dagger]$ and $[a_{(\mathrm{G})}, a_{(\mathrm{L})}^\dagger]$, and since the $a_{(\mathrm{L})}^\dagger$'s are absent, the norm of any such state is positive semi-definite

$$\langle \Psi|\Psi\rangle \geq 0. \tag{6.47}$$

The modes $a_{(1,2)}^\dagger$ have a positive norm while $a_{(\mathrm{G})}^\dagger$ is null.

---

[6]There is a lot of arbitrariness in defining the polarisation vectors for each momentum p, but it does not matter.

[7]The crossing between L and G is due to the construction of the basis using two light-like directions.

[8]Note that $a_{(\mathrm{G})}$ commutes with $a_{(\mathrm{G})}^\dagger$ and $a_{(1,2)}^\dagger$ but not with $a_{(\mathrm{L})}^\dagger$.

Null States. Consider a physical state $|\Psi\rangle$ which contains an excitation of type $a^\dagger_{(G)}$, i.e. a state which can be written as

$$|\Psi\rangle = a^\dagger_{(G)}(\vec{p})|\Omega\rangle \qquad (6.48)$$

with some other physical state $|\Omega\rangle$. This state has zero norm by the physical state condition

$$\langle\Psi|\Psi\rangle = \langle\Omega|a_{(G)}(\vec{p})a^\dagger_{(G)}(\vec{p})|\Omega\rangle = \langle\Omega|a^\dagger_{(G)}(\vec{p})a_{(G)}(\vec{p})|\Omega\rangle = 0. \qquad (6.49)$$

Null states are not normalisable and therefore have to be interpreted appropriately. Typically null states are irrelevant because QM is a probabilistic framework. Something that takes place with probability zero does not happen. Nevertheless, some consistency requirements have to be fulfilled:

By the same argument as above, we can show that a null state

$$|\Psi\rangle = a^\dagger_{(G)}(\vec{p})|\Omega\rangle \qquad (6.50)$$

actually has vanishing scalar products with any physical state $|\Phi\rangle$ due to the physicality condition of the latter

$$\langle\Phi|\Psi\rangle = \langle\Phi|a^\dagger_{(G)}(\vec{p})|\Omega\rangle = 0. \qquad (6.51)$$

In particular, this implies that the sum $|\Psi'\rangle$ of a physical state $|\Psi\rangle$ and some null state

$$|\Psi'\rangle = |\Psi\rangle + a^\dagger_{(G)}(\vec{p})|\Phi\rangle \qquad (6.52)$$

behaves just like the original physical state $|\Psi\rangle$ in scalar products

$$\langle\Phi|\Psi'\rangle = \langle\Phi|\Psi\rangle + \langle\Phi|a^\dagger_{(G)}(\vec{p})|\Omega\rangle = \langle\Phi|\Psi\rangle. \qquad (6.53)$$

We should thus impose an equivalence relation on the physical Fock space

$$|\Psi\rangle \simeq |\Psi'\rangle = |\Psi\rangle + a^\dagger_G(\vec{p})|\Omega\rangle. \qquad (6.54)$$

Any two states which differ by a state which is in the image of some $a^\dagger_{(G)}$ are physically equivalent. In other words, physical states of the gauge field are not described by particular vectors but by equivalence classes of vectors.

We may use states which have no contribution of $a_{(G)}$ as reference states of the equivalence classes[9]

$$|\Psi\rangle = a^\dagger_{(1,2)} \cdots a^\dagger_{(1,2)}|0\rangle. \qquad (6.55)$$

These representatives show that we have two states for each momentum $\vec{p}$. It matches nicely with the massless UIR's of the Poincaré group with positive and negative helicity $h = \pm 1$. The particle excitations of the electromagnetic field are the photons.

---

[9]Although this appears to be a useful choice at first sight, it is not at all unique. By a change of basis for the polarisation vectors at any given $\vec{p}$ we can add any amount of $a^\dagger_{(G)}$ to $a^\dagger_{(1,2)}$. The new states are certainly in the same equivalence class, but they are different representatives.

Gauge Transformations. However, inserting some gauge potentials $A_\mu(x)$ into the scalar product may actually lead to some dependence on null states. Let us therefore compute

$$\langle\Phi|A_\mu(x)|\Psi'\rangle = \langle\Phi|A_\mu(x)|\Psi\rangle + \langle\Phi|A_\mu(x)a^\dagger_{(G)}(\vec{p})|\Omega\rangle$$
$$= \langle\Phi|A_\mu(x)|\Psi\rangle + \langle\Phi|[A_\mu(x), a^\dagger_{(G)}(\vec{p})]|\Omega\rangle \tag{6.56}$$

We write $a^\dagger_{(G)}(\vec{p}) = p{\cdot}a^\dagger(\vec{p})$ and the commutator evaluates to

$$[A_\mu(x), a^\dagger_{(G)}(\vec{p})] = p_\mu e^{-ip\cdot x} = i\partial_\mu e^{-ip\cdot x} \tag{6.57}$$

The expectation value of $A_\mu(x)$ thus changes effectively by a derivative term

$$A_\mu(x) \mapsto A_\mu(x) + \frac{\langle\Phi|\Omega\rangle}{\langle\Phi|\Psi\rangle} i\partial_\mu e^{-ip\cdot x}. \tag{6.58}$$

This is just a gauge transformation of the potential $A_\mu(x)$. We observe that the states $|\Psi\rangle$ and $|\Psi'\rangle$ lead to two expectation values which differ by a gauge transformation of the fields within the expectation value. Note that the gauge transformation does not leave the Lorenz gauges

$$[\partial{\cdot}A(x), a^\dagger_{(G)}(\vec{p})] = i\partial^2 e^{-ip\cdot x} = 0. \tag{6.59}$$

Hence null states induce residual gauge transformation within the Lorenz gauges.

Now it appears that the choice of representative in an equivalence class has undesirable impact on certain expectation values. Gladly, this does not apply to gauge-invariant observables. For instance, the electromagnetic field strength is unaffected

$$[F_{\mu\nu}(x), a^\dagger_{(G)}(\vec{p})] = \partial_\mu(p_\nu e^{-ip\cdot x}) - \partial_\nu(p_\mu e^{-ip\cdot x}) = 0. \tag{6.60}$$

Moreover, the coupling of the gauge potential to a conserved current J^μ

$$J[A] = \int d^4x\, J^\mu(x)A_\mu(x) \tag{6.61}$$

commutes with $a^\dagger_{(G)}$

$$[J[A], a^\dagger_{(G)}(\vec{p})] = i\int d^4x\, J^\mu(x)\partial_\mu e^{-ip\cdot x}$$
$$= -i\int d^4x\, e^{-ip\cdot x}\partial_\mu J^\mu(x) = 0. \tag{6.62}$$

The expectation value of any gauge-invariant operator composed from $F_{\mu\nu}$, $J[A]$ or similar combinations thus does not depend on the choice of representatives, and it is consistent to define physical states as equivalence classes.

6.4   Casimir Energy

At this point we can already compute a quantum effect of the electromagnetic field, the Casimir effect. The Casimir effect is a tiny force between nearby conductors which exists even in the absence of charges, currents or medium. In classical electrodynamics no forces are expected. There are several alternative explanations for the quantum origin of the force. One is the exchange of virtual photons between the conductors. An equivalent explanation attributes the force to a change of vacuum energy of the electromagnetic field induced by the presence of the plates. The latter one has a quite efficient derivation, and we shall present it here.

Setup.   We place two large planar metal plates at a small distance into the vacuum (much smaller than their size, but much larger than atomic distances). In our idealised setup, the plates extend infinitely along the x and y directions. They are separated by the distance a in the z direction. We will not be interested in the microscopic or quantum details of the metal objects. We simply assume that they are classical conductors and that they shield the electromagnetic field efficiently.
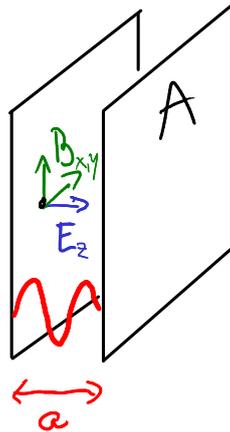
(6.63)

At the surface of the plates, the electric fields must be orthogonal $E_x = E_y = 0$ while the magnetic field must be parallel $B_z = 0$. In order to match these conditions simultaneously at both plates, the z-component of the wave vector (momentum) must be quantised

$$p_z \in \frac{\pi}{a}\, \mathbb{Z}. \tag{6.64}$$

Careful analysis shows that for $p_z = 0$ only one of the two polarisation vectors is permissible. Conversely, for $p_z \neq 0$ both polarisations are good. To achieve cancellations in this case, each wave must be synchronised to its reflected wave where $p_z \to -p_z$. Hence we should only count the contributions with $p_z > 0$.

Vacuum Energy.   Just like the scalar field, the electromagnetic field carries some vacuum energy. The discretisation modifies the vacuum energy E_0, which results in a force between the plates if the new vacuum energy depends on the distance a.

The sum and integral of all permissible modes between the plates yields the energy

E per area A [10]

$$E = \int \frac{A \, dp_x \, dp_y}{(2\pi)^2} \left(\tfrac{1}{2} e(p_x, p_y, 0) + 2 \sum_{n=1}^{\infty} \tfrac{1}{2} e(p_x, p_y, \pi n/a) \right) \qquad (6.65)$$

For convenience we shall exploit the rotation symmetry in the x-y-plane to simplify the expression to

$$\frac{E}{A} = \int_0^{\infty} \frac{p \, dp}{2\pi} \left(\tfrac{1}{2} p + \sum_{n=1}^{\infty} \sqrt{p^2 + \pi^2 n^2/a^2} \right) \qquad (6.66)$$

As discussed earlier, this expression diverges due to UV contributions at large momenta.

Regularisation. We also emphasised earlier that infinities are largely our own fault. The idealised setup was somewhat too ideal.

For macroscopic electromagnetic waves, we certainly made the right assumption of total reflection. But it is also clear that the conducting plates will behave differently for hard gamma radiation. This is precisely where the problems arise, so we seem to be on the right track. Electromagnetic waves with wave length much smaller than atomic distances or energies much larger than atomic energy levels will pass the conducting plates relatively unperturbed. These modes therefore should be discarded from the above sum.[11]

Therefore, we must introduce a UV cutoff for the modes. Define a function $f(e)$ which is constantly 1 for sufficiently small energies, constantly 0 for sufficiently large energy and which somehow interpolates between the 1 and 0 for intermediate energy. The cutoff replaces each contribution $\tfrac{1}{2} e$ by $\tfrac{1}{2} f(e) e$

$$\frac{E_{\text{IR}}}{A} = \int_0^{\infty} \frac{p \, dp}{2\pi} \left(\tfrac{1}{2} p f(p) + \sum_{n=1}^{\infty} \sqrt{p^2 + \pi^2 n^2/a^2} \, f(\sqrt{\ldots}) \right). \qquad (6.67)$$

Let us keep in mind the remaining contribution in the ultraviolet where we convert the sum to an integral due to the absence of quantisation in the z-direction

$$\frac{E_{\text{UV}}}{A} = \int_0^{\infty} \frac{p \, dp}{2\pi} \int_0^{\infty} dn \sqrt{p^2 + \pi^2 n^2/a^2} \left(1 - f(\sqrt{\ldots}) \right). \qquad (6.68)$$

---

[10] A sum over the modes in some box of volume V in d dimensions turns into an integral over momenta when the volume is very large. In a box, the positive and negative modes are coupled, so the integral is over positive p only with integration measure $V \, d^d p/\pi^d$. In the absence of boundary contributions, the integration domain extends to positive and negative p which is compensated by the measure $V \, d^d p/(2\pi)^d$.

[11] The modes do contribute to the vacuum energy between the plates. Importantly, the distance between the plates will hardly enter their contribution, and consequently they cannot contribute to forces.

Summation. This expression is now finite, but certainly depends on the cutoff in $f(e)$. Let us write it as a sum of integrals

$$\frac{E_{\mathrm{IR}}}{A} = \tfrac{1}{2} F(0) + \sum_{n=1}^{\infty} F(n), \tag{6.69}$$

with

$$F(n) = \int_0^{\infty} \frac{p\,dp}{2\pi} \sqrt{p^2 + \pi^2 n^2/a^2}\, f\big(\sqrt{p^2 + \pi^2 n^2/a^2}\big) \tag{6.70}$$

It is convenient to use energy as integration variable

$$e = \sqrt{p^2 + \pi^2 n^2/a^2}, \qquad p\,dp = e\,de, \tag{6.71}$$

and write the integral as

$$F(n) = \frac{1}{2\pi} \int_{\pi n/a}^{\infty} de\, e^2 f(e). \tag{6.72}$$

The Euler–MacLaurin summation formula writes the above sum for E_{IR}/A as an integral plus correction terms

$$\frac{E_{\mathrm{IR}}}{A} = \int_0^{\infty} dn\, F(n) - \sum_{k=1}^{\infty} (-1)^k \frac{\mathrm{B}_{2k}}{(2k)!} F^{(2k-1)}(0), \tag{6.73}$$

where we have used that the function $F(n)$ is constantly zero at ∞ due to the cutoff. Here B_n is the n-th Bernoulli number.

Let us analyse the two terms: The first term we can rewrite as

$$\frac{E_{\mathrm{int}}}{A} = \frac{1}{2\pi} \int_0^{\infty} dn \int_{\pi n/a}^{\infty} de\, e^2 f(e) = \frac{2a}{4\pi^2} \int_0^{\infty} de' \int_{e'}^{\infty} de\, e^2 f(e). \tag{6.74}$$

It depends on the cutoff, but it is manifestly linear in a. In fact, it combines nicely with the contribution from UV modes above the cutoff that we dropped earlier

$$\frac{E_0}{A} = \frac{E_{\mathrm{UV}} + E_{\mathrm{int}}}{A} = \frac{2a}{4\pi^2} \int_0^{\infty} de' \int_{e'}^{\infty} de\, e^2. \tag{6.75}$$

As such it represents the vacuum energy of the enclosed volume in the absence of plates. The same vacuum energy density is present outside the plates.[12] This term therefore does not contribute to the force because any shift of the plate would merely transfer some vacuum energy from the inside to the outside leaving the overall energy invariant. The fact that E_0 is formally infinite does not play a role. We therefore consider only the change in energy $\Delta E = E - E_0$ arising from the second term of the Euler–MacLaurin summation.

The second term can be evaluated near $n = 0$

$$F(n) = F(0) - \frac{1}{2\pi} \int_0^{\pi n/a} de\, e^2 f(e) = F(0) - \frac{\pi^2 n^3}{6a^3}, \tag{6.76}$$

---

[12]The factor of $2/2\pi$ is interpreted as follows: $1/2\pi$ is the correct measure for integration over p_z. Moreover, in the factor of 2 compensates for the restricted integration region $p_z \geq 0$.

where we used that $f(e) = 1$. Quite surprisingly, $F(n)$ is a polynomial with two terms. All cutoff dependence is in $F(0)$ which does not appear in the summation formula.[13] The single correction term contributes the following vacuum energy $(B_4 = -1/30)$

$$\frac{\Delta E}{A} = -\frac{B_4}{4!} F^{(3)}(0) = -\frac{\pi^2}{720a^3} .$$
(6.77)

The presence of the conducting plates decreases the vacuum energy by some amount proportional to $1/a^3$.

Casimir Force. The Casimir force can be expressed as the pressure

$$P = \frac{F}{A} = \frac{\Delta E'(a)}{A} = \frac{\pi^2}{240a^4} .$$
(6.78)

Some properties:

- Bringing the plates closer decreases the energy, hence the Casimir force is attractive.
- It increases with the fourth power of the inverse distance as the plates come closer.
- It is a quantum effect, and there are hidden factor of $\hbar$ and c. Due to the fourth-power behaviour it can nevertheless be detected at reasonable separations. It becomes relevant at micrometer distance.
- It does not depend on the coupling strength of the electromagnetic field or on the elementary charge.

6.5 Massive Vector Field

So far we have discussed the massless vector field. Among the UIR's of the Poincaré group there is also the massive representation with spin 1. Massive vector particles exist in nature as the $W_\pm$ and Z_0 bosons transmitting the weak nuclear interactions.[14]

Lagrangian. We can add a mass term to the vector Lagrangian to obtain the corresponding quantum field

$$\mathcal{L} = -\tfrac{1}{2}\partial_\mu V_\nu \partial^\mu V^\nu + \tfrac{1}{2}\partial_\mu V_\nu \partial^\nu V^\mu - \tfrac{1}{2}m^2 V^\mu V_\mu.$$
(6.79)

The corresponding equation of motion reads

$$\partial^2 V_\mu - \partial_\mu \partial^\nu V_\nu - m^2 V_\mu = 0.$$
(6.80)

---

[13]The reason is apparently that the cutoff is in a region of energies where the difference between a sum and an integral does not matter.

[14]The implementation of interacting massive vector fields actually needs much more care. Interacting vectors fields can acquire mass only through the Higgs mechanism.

By taking the total derivative of this equation, we see that it implies the simpler equation $-m^2 \partial^\mu V_\mu = 0$. Substituting this result in the original equation of motion then yields a system of two equations

$$\partial^2 V_\mu - m^2 V_\mu = 0, \qquad \partial^\mu V_\mu = 0. \tag{6.81}$$

The first equation is the Klein–Gordon equation for each component of V_μ, the second equation removes one of the four potential orientations. The degrees of freedom agree with the classification of UIR's.

Correlators. We now want to quantise this system. In the canonical approach we first derive the conjugate momenta

$$\Pi_\mu = \dot{V}_\mu - \partial_\mu V_0. \tag{6.82}$$

As before, there is no conjugate momentum for the field V_0 hinting at the presence of constraints. Constrained systems are somewhat tedious to handle in the Hamiltonian framework and therefore in canonical quantisation. Instead, let us take a shortcut. We consider the fields to be operators and cook up unequal-time commutation relations

$$[V_\mu(x), V_\nu(y)] = \Delta^{\mathrm{V}}_{\mu\nu}(x - y). \tag{6.83}$$

Our previous experience has shown that correlators can be composed from derivatives acting on the correlator of the scalar field. This automatically implements the Klein–Gordon equation. Here we propose[15] [16]

$$\Delta^{\mathrm{V}}_{\mu\nu}(x) = \left(\eta_{\mu\nu} - m^{-2} \partial_\mu \partial_\nu \right) \Delta(x). \tag{6.84}$$

The combination of derivatives was constructed such that Δ^{V} satisfies the polarisation equations

$$\partial^\mu \Delta^{\mathrm{V}}_{\mu\nu}(x) = \partial^\nu \Delta^{\mathrm{V}}_{\mu\nu}(x) = 0. \tag{6.85}$$

Equal-Time Commutators. Next let us see what this proposal implies for the equal-time commutators. The non-vanishing ones read as follows

$$
\begin{aligned}
[V_0(\vec{x}), V_k(\vec{y})] &= im^{-2} \partial_k \delta^3(\vec{x} - \vec{y}), \\
[V_0(\vec{x}), \dot{V}_0(\vec{y})] &= -im^{-2} \partial_k \partial_k \delta^3(\vec{x} - \vec{y}), \\
[V_k(\vec{x}), \dot{V}_l(\vec{y})] &= i\delta_{kl} \delta^3(\vec{x} - \vec{y}) - im^{-2} \partial_k \partial_l \delta^3(\vec{x} - \vec{y}), \\
[\dot{V}_0(\vec{x}), \dot{V}_k(\vec{y})] &= i\partial_k \delta^3(\vec{x} - \vec{y}) - im^{-2} \partial_k \partial_l \partial_l \delta^3(\vec{x} - \vec{y}).
\end{aligned}
\tag{6.86}
$$

These relations appear somewhat unusual since they mix time and space components of V_μ.

---

[15] The correlator Δ_+ and the propagator G take an equivalent form in terms of their scalar field counterparts.

[16] The factor of $1/m^2$ in not as innocent as it may appear. When adding interactions, this term involving an inverse mass scale actually makes the theory behave badly for large momenta.

Let us replace the time derivatives $\dot{V}_k$ by the associated conjugate momenta Π_k. For the spatial components we recover the canonical commutator

$$[V_k(\vec{x}), \Pi_l(\vec{y})] = i\delta_{kl}\delta^3(\vec{x} - \vec{y}). \tag{6.87}$$

The commutators involving V_0 and $\dot{V}_0$ can be recovered using the equations of motion. The latter actually give an explicit solution for the field V_0 and its time derivative $\dot{V}_0$

$$V_0 = -m^{-2}\partial_k\Pi_k, \qquad \dot{V}_0 = \partial_k V_k. \tag{6.88}$$

In other words, V_0 is not an elementary field and its commutation relations follow from the canonical one above

$$[\dot{V}_0(\vec{x}), \Pi_k(\vec{y})] = i\partial_k\delta^3(\vec{x} - \vec{y}),$$
$$[V_k(\vec{x}), V_0(\vec{y})] = im^{-2}\partial_k\delta^3(\vec{x} - \vec{y}),$$
$$[V_0(\vec{x}), \dot{V}_0(\vec{y})] = im^{-2}\partial_k\partial_k\delta^3(\vec{x} - \vec{y}). \tag{6.89}$$

Hamiltonian Framework. We have obtained a reasonable QFT framework for our massive scalar field. Now we can revisit the Hamiltonian framework. First we perform a Legendre transformation of the Lagrangian for spatial components of the fields V_k [17]

$$H = \int d^3\vec{x} \left(\Pi_k \dot{V}_k - \mathcal{L} \right)$$
$$= \int d^3\vec{x} \left(\tfrac{1}{2}\Pi_k\Pi_k + \tfrac{1}{2}m^{-2}\partial_k\Pi_k\partial_l\Pi_l \right.$$
$$\left. + \tfrac{1}{2}\partial_k V_l\partial_k V_l - \tfrac{1}{2}\partial_l V_k\partial_k V_l + \tfrac{1}{2}m^2 V_k V_k \right). \tag{6.90}$$

Here, we have also substituted the solution for the field V_0 and its time derivative.

We note that the Hamiltonian is slightly unusual in that it contains derivatives of the momenta along with inverse powers of the mass. The inverse powers of the mass in fact prevent us from taking the massless limit.[18]

Gladly, this Hamiltonian implies the desired equations of motion

$$\dot{V}_k = -\{H, V_k\} = \Pi_k - m^{-2}\partial_k\partial_l\Pi_l,$$
$$\dot{\Pi}_k = -\{H, \Pi_k\} = \partial_l\partial_l V_k - \partial_k\partial_l V_l - m^2 V_k. \tag{6.91}$$

It is not at all obvious that these equations imply the Klein–Gordon equation. However, their twisted form is required to be able to solve for the field V_0 easily and thereby obtain the correct energy.

---

[17]The Hamiltonian is manifestly positive since $\tfrac{1}{2}\partial_k V_l\partial_k V_l - \tfrac{1}{2}\partial_l V_k\partial_k V_l = \tfrac{1}{4}(\partial_k V_l - \partial_l V_k)^2$.

[18]We may impose a gauge by demanding $\partial_k\Pi_k = -m^{-2}V_0 = 0$. This eliminates the inverse mass from the Hamiltonian and validates the massless limit. Using $\partial_k\Pi_k = -m^{-2}V_0$ the gauge also implies $\dot{V}_0 = \partial_k V_k = 0$, i.e. the gauge is the Coulomb gauge.

7 Interactions

We have learned a lot about the three basic constituents of QFT in four dimensions:

- scalar fields (spin $j = 0$ or helicity $h = 0$),
- spinor fields (spin $j = \frac{1}{2}$ or helicity $h = \pm\frac{1}{2}$),
- vector fields (helicity $h = \pm 1$ or spin $j = 1$).

So far we considered only free fields. The particle number was conserved by all processes and most operators.

Now we would like to introduce interactions between such fields. Unfortunately, interactions cannot be treated exactly.

We have to assume the strength of interactions to be sufficiently small. The well-understood free fields will dominate, and we insert interactions as small perturbations. This eventually leads us to Feynman diagrams to describe particle interactions order by order.

7.1 Interacting Lagrangians

One of the main reasons to consider QFT is its ability to deal with processes that do not conserve the number of particles.

Quantum fields are particle creation and annihilation operators: $\phi \to a, a^\dagger$. So far, we used them for two purposes:

- to build the multi-particle Fock space from a vacuum state, i.e. $a^\dagger \ldots a^\dagger |0\rangle$;
- to write conserved charges as quadratic combinations of the fields which conspired to yield one creation and one annihilation operator $a^\dagger a$ and thus conserve the particle number.

Combining more than two fields typically yields a quantum operator which changes the particle number.

Time evolution of a quantum system is governed by its Hamiltonian, therefore it is natural to include such higher-order terms in it and consequently in the Lagrangian.

Scalar Interactions. For a real scalar field we could consider an interacting Lagrangian of the form

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu\phi\partial_\mu\phi - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{6}\mu\phi^3 - \tfrac{1}{24}\lambda\phi^4. \tag{7.1}$$

This model called the ϕ^4 theory.[1] It is perhaps the conceptually simplest interacting QFT model, but it leads to very non-trivial physics. The ϕ^4 term is also an interaction of the scalar Higgs field which is essential for the Higgs mechanism.

We might also add higher-order terms or terms involving derivatives such as

$$\phi^5, \quad \phi(\partial\phi)^2, \quad \phi^2(\partial\phi)^2, \quad (\partial\phi)^4, \quad \dots . \tag{7.2}$$

Such terms are in principle allowable in QFT, but they have some undesirable features to be discussed later. We will, however, never add non-local terms of the type to the action

$$\int d^4x\, \phi(x)\phi(x+a), \qquad \int d^4x\, d^4y\, f(x,y)\phi(x)\phi(y). \tag{7.3}$$

These terms represent some unphysical action at a distance, we consider only local interactions which can be written using a local Lagrangian[2]

$$S = \int d^4x\, \mathcal{L}(x), \qquad \mathcal{L}(x) = \mathcal{L}[\phi(x), \partial\phi(x), \dots]. \tag{7.4}$$

The equation of motion for the above Lagrangian reads

$$\partial^2\phi - m^2\phi - \tfrac{1}{2}\mu\phi^2 - \tfrac{1}{6}\lambda\phi^3 = 0. \tag{7.5}$$

It is a non-linear differential equation. Our usual strategy to deal with the differential equation of motion was to go to momentum space

$$0 = -p^2\phi(p) - m^2\phi(p) - \tfrac{1}{2}\mu \int \frac{d^4q}{(2\pi)^4}\, \phi(q)\phi(p-q)$$
$$- \tfrac{1}{6}\lambda \int \frac{d^4q_1\, d^4q_2}{(2\pi)^8}\, \phi(q_1)\phi(q_2)\phi(p-q_1-q_2). \tag{7.6}$$

Unfortunately, we obtain an integral equation instead of an algebraic equation. We cannot solve it in general, but for small $\mu \ll m$ and small $\lambda \ll 1$ we can try to find useful approximations.

Quantum Electrodynamics. Electrons and positrons carry an electrical charge $\mp q$ and their conserved current $J^\mu = -q\bar\psi\gamma^\mu\psi$ couples to the Maxwell equations. Putting together the Dirac and Maxwell Lagrangians we can simply add a source term for the electromagnetic potential $J^\mu A_\mu$

$$\mathcal{L}_{\text{QED}} = \bar\psi(i\gamma^\mu\partial_\mu - m)\psi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + q\,\bar\psi\gamma^\mu\psi\, A_\mu. \tag{7.7}$$

---

[1]One often drops the term ϕ^3 and gains a discrete symmetry $\phi \mapsto -\phi$. A term ϕ^3 without a term ϕ^4 would lead to a potential unbounded from below.

[2]Non-local terms could be recovered as Taylor series involving derivatives of arbitrary order. It is therefore desirable to restrict the number of derivatives in $\mathcal{L}$.

This yields the desired inhomogeneous Maxwell equations, but also a modification of the Dirac equation

$$\partial_\mu F^{\mu\nu} = -q\bar{\psi}\gamma^\nu\psi.$$
$$(\partial^\mu\gamma_\mu - m)\psi = -q\gamma^\mu A_\mu\psi. \tag{7.8}$$

The above model is called quantum electrodynamics (QED). It is a model that has been tested at a remarkable accuracy within its domain of validity, i.e. at low energies where the other elementary particles play no essential role. For instance, the electron anomalous magnetic dipole moment, also known as $g - 2$, was predicted to more than 10 digits, and many of the leading digits are due to plain QED alone.

In the Standard Model, the above type of interaction between vectors and spinors is arguably the most important one because it couples matter in the form of spinor fields (leptons and quarks) to forces in the form of vector fields (photons, gluons and others).

Gauge Invariance. A crucial property of the electromagnetic potential is its gauge symmetry. In the quantisation procedure it eliminated an unphysical degree of freedom of the electromagnetic potential. We therefore want to preserve this symmetry in the presence of interactions.

The interaction term breaks the original gauge symmetry, but the latter can be restored by extending the symmetry to the Dirac field

$$A'_\mu(x) = A_\mu(x) + \partial_\mu\alpha,$$
$$\psi'(x) = \exp\big(iq\alpha(x)\big)\psi(x). \tag{7.9}$$

Note that the latter transformation rule is just the global U(1) symmetry of the Dirac field which is responsible for conservation of the current J^μ. This global symmetry is enhanced to a local transformation parameter $\alpha(x)$. The derivative terms of $\alpha(x)$ are now compensated by the inhomogeneous gauge transformation of the potential A_μ.

There is a construction which makes the gauge invariance more manifest. Introduce the gauge covariant derivative

$$D_\mu = \partial_\mu - iqA_\mu. \tag{7.10}$$

Under gauge transformations this operator transforms homogeneously

$$\begin{aligned} D'_\mu &= \partial_\mu - iqA'_\mu = \partial_\mu - iqA_\mu - iq\partial_\mu\alpha \\ &= D_\mu + [D_\mu, -iq\alpha] \\ &= \exp(+iq\alpha)D_\mu\exp(-iq\alpha). \end{aligned} \tag{7.11}$$

In the QED Lagrangian written with a covariant derivative

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{7.12}$$

the factors of $\exp(\pm iq\alpha)$ trivially cancel between $\bar{\psi}$, D_μ and ψ. Moreover the electromagnetic field strength can be written as

$$F_{\mu\nu} \sim [D_\mu, D_\nu], \tag{7.13}$$

which makes manifest its invariance under gauge transformations.

Consequently, we can also couple the complex scalar field to the electromagnetic field via its kinetic term

$$\mathcal{L}_{\text{SQED}} = -(D^\mu\phi)^* D_\mu\phi - m^2|\phi|^2 - \tfrac{1}{4}\lambda|\phi|^4 - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}. \tag{7.14}$$

This model is called scalar QED.

Further Interactions. Let us list some other simple interactions. We want to consider only those interactions which respect Lorentz symmetry. Curiously all of them appear in the Standard Model.

Two Dirac spinors can be multiplied to form a scalar combination. This can be multiplied by a scalar field

$$\bar{\psi}\psi\,\phi. \tag{7.15}$$

This term was originally proposed by Yukawa for the interaction between nucleons of spin $\tfrac{1}{2}$ and scalar pions. In the standard model such term couple the Higgs field to the leptons and quarks.

A similar term coupling a scalar and two Dirac fermions but with different parity properties is

$$\bar{\psi}i\gamma^5\psi\,\phi. \tag{7.16}$$

There is also an analog of the spinor-vector coupling with different parity properties

$$\bar{\psi}\gamma^5\gamma^\mu\psi\,A_\mu. \tag{7.17}$$

This axial vector coupling term is relevant to the weak nuclear interactions. Here, gauge invariance needs to extend to local chiral transformations of the spinors.

The above interactions for Dirac 4-spinors can be written in terms of more elementary chiral 2-spinor fields,

$$\chi^{\mathrm{T}}\sigma^2\chi\,\phi, \qquad \chi^\dagger\bar{\sigma}^\mu\chi\,A_\mu. \tag{7.18}$$

Note that the first interaction is complex, and therefore only some real projection can appear in the Lagrangian. This leads to two couplings, one for the real part and one for the imaginary part. The second term is perfectly real and requires a single real coupling constant.

Power Counting. We have encountered several types of interaction terms. These have a rather simple form with very few factors. Moreover, most of the simple terms have been observed directly or indirectly in nature. However, there

are many more local terms one could imagine, but which have not been observed. What distinguishes the above interactions?

To answer this question, consider the mass dimension. The action S must be a dimensionless quantity.[3] The action is the integral of the Lagrangian $S = \int d^4x\, \mathcal{L}$ and length counts as inverse mass, $dx \sim m^{-1}$, therefore the Lagrangian must have mass dimension 4,

$$\mathcal{L} \sim m^4. \tag{7.19}$$

The kinetic terms $(\partial\phi)^2$, $\bar{\psi}\partial\psi$ and F^2 where the derivative counts as a mass, $\partial \sim m$, determine the mass dimensions of the scalar, spinor and vector fields

$$\phi \sim A_\mu \sim m, \qquad \psi \sim m^{3/2}. \tag{7.20}$$

The mass dimension of the remaining terms is now fixed, e.g. for the mass terms

$$\phi^2 \sim m^2, \qquad \bar{\psi}\psi \sim m^3, \tag{7.21}$$

and for the simple interaction terms

$$\phi^3 \sim m^3, \qquad \phi^4 \sim \bar{\psi}\gamma^\mu\psi A_\mu \sim \bar{\psi}\psi\phi \sim m^4. \tag{7.22}$$

All of these terms have mass dimension at most 4. When they appear in the Lagrangian $\mathcal{L} \sim m^4$, their coupling constant must compensate for the missing mass dimension. The scalar and fermion mass terms therefore read $m^2\phi^2$, $m\bar{\psi}\psi$. Among the interaction terms, only ϕ^3 requires a dimensionful coupling $\mu \sim m$. All the other terms have mass dimension 4 and their coupling constants are plain numbers.

We can take the bound of mass dimension 4 [4] as an experimentally observed principle. There are good reasons to consider only terms of this type:

- Such interactions are reasonably simple.
- There are only finitely many such term, hence finitely many parameters for the model.
- All higher-dimensional terms require a coupling constant with negative mass dimension.
- Coupling constants with negative mass dimension lead to undesirable effects in the ultraviolet or short-distance regimes.
- Such theories are called non-renormalisable. Renormalisability will be considered later in QFT II.
- In the infrared or long-distance regime,[5] only the interactions of mass dimension up to 4 are relevant. The higher-dimensional terms have small effects are are mostly irrelevant.

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[3]Quantities that appear in an exponent must be dimensionless numbers. The action carries the same units as Planck constant $\hbar$ which in natural units is a number $\hbar = 1$.

[4]More generally, the number of spacetime dimensions.

[5]The meaning of long-distance depends on the point of view. It can be astronomical units, everyday length scales, atomic scales or even less when interested in fundamental description of nature.

- The mass term for a vector field appears as an inverse power in the massive vector propagator. This also leads to a non-renormalisable model. To explain the mass of the W and Z vector bosons we rely on the Higgs mechanism.[6]

Symmetries. Unfortunately, it is generally hard to extract information from interacting QFT models. Usually we can only do certain approximations. Symmetries are powerful concepts in QFT because they can apply to interacting models as well:

- Free theories are somewhat trivial because there are infinitely many conservation laws. For example, the particle number is conserved, but it is related only to a non-local transformation.
- Only few of the conservation laws typically survive when interactions are added. Those are related to global symmetries preserved by the interactions.
- The conservation laws allow to make certain statements on the result of QFT observables even when computations are not feasible.
- Symmetries of the classical theory are not necessarily respected by the quantum theory. Such symmetries are called anomalous.
- Anomalies of local symmetries are typically bad because they spoil gauge redundancies which are required for consistency.

For example, consider ϕ^4 theory with a complex field given by the Lagrangian

$$\mathcal{L} = -\partial^\mu \phi^* \partial_\mu \phi - m^2 |\phi|^2 - \tfrac{1}{4}\lambda |\phi|^4. \tag{7.23}$$

It is invariant under global multiplication by a complex phase $\phi \to e^{i\alpha}\phi$. This leads to the same conserved current as for the complex scalar

$$J^\mu = -i(\partial^\mu \phi^* \phi - \phi^* \partial^\mu \phi). \tag{7.24}$$

The associated current $Q = N_a - N_b$ is exactly conserved even in the presence of interactions. Conservation of the individual number operators N_a and N_b, however, is broken by interactions.

7.2 Interacting Field Operators

Consider an interacting field theory whose fields (and conjugate momenta) we will collectively denote by $\phi(x)$. More concretely, we can consider ϕ^4 theory.

We want to compute some correlation function, for example a correlator of two fields at different times t_1, t_2

$$F(x_2, x_1) = \langle 0|\phi(t_2, \vec{x}_2)\phi(t_1, \vec{x}_1)|0\rangle. \tag{7.25}$$

---

[6]Also known as the Englert–Brout–Higgs–Guralnik–Hagen–Kibble–Anderson–and–perhaps–also–'t–Hooft mechanism.

Interacting Field. As before, we can quantise the field ϕ on a time slice at some time t_0. This step is equivalent to a free field ϕ because the Poisson brackets are the same.

The full time dependence of ϕ is recovered by conjugating with the Hamiltonian

$$\phi(t, \vec{x}) = \exp(iH(t - t_0))\phi(\vec{x})\exp(iH(t_0 - t)). \tag{7.26}$$

Supposing that the vacuum is time-invariant,[7] we can write the correlator as

$$F(x_2, x_1) = \langle 0|\phi(\vec{x}_2)\exp(-iH(t_2 - t_1))\phi(\vec{x}_1)|0\rangle. \tag{7.27}$$

Now everything is explicitly known except how to exponentiate H in practice. The latter is a hard problem.

Interaction Picture. We can do slightly better whenever the interactions are weak. In this case, the dominant contribution should come from the free Hamiltonian H_0. The quantisation of fields at a given time slice is the same. We can thus identify the fields at time t_0

$$\phi_0(\vec{x}) = \phi(\vec{x}). \tag{7.28}$$

Time evolution of the free field ϕ_0 is governed by the free Hamiltonian H_0

$$\phi_0(t, \vec{x}) = \exp(iH_0(t - t_0))\phi(\vec{x})\exp(iH_0(t_0 - t)). \tag{7.29}$$

We know almost everything about this field. For weak interactions and small times $t \simeq t_0$, we expect the free field $\phi_0(t, \vec{x})$ to be a suitable approximation for the full field $\phi(t, \vec{x})$.

Comparing ϕ to ϕ_0 we can write

$$\phi(t, \vec{x}) = U(t, t_0)^{-1}\phi_0(t, \vec{x})U(t, t_0) \tag{7.30}$$

with the time evolution operator

$$U(t, t_0) = \exp\big(iH_0[\phi_0](t - t_0)\big)\exp\big(iH[\phi_0(t_0)](t_0 - t)\big). \tag{7.31}$$

For small interactions and small times, this operator is approximately the identity.

This is called the interaction picture, it is a mixture between the Schrödinger and the Heisenberg pictures

- In the Schrödinger picture, the field is defined on a constant time slice $\phi = \phi(t_0)$ and the operator $\exp(i(t - t_0)H)$ evolves states in time.
- In the Heisenberg picture, the field $\phi(t)$ carries the full time dependence, there is no need for a time evolution operator.
- In the interaction picture, the field $\phi_0(t)$ carries the time dependence of a free particle and the operator $U(t, t_0)$ evolves states in time.

---

[7]A constant energy of the vacuum can always be eliminated by subtracting it from H.

The correlator in question becomes

$$\langle 0|U(t_2,t_0)^{-1}\phi_0(t_2,\vec{x}_2)U(t_2,t_0)U(t_1,t_0)^{-1}\phi_0(t_1,\vec{x}_1)U(t_1,t_0)|0\rangle. \qquad (7.32)$$

Note that products of time evolution operators can be joined in the obvious fashion, they form a group[8]

$$
\begin{aligned}
& U(t_2,t_1)U(t_1,t_0) \\
&= \exp(iH_0[\phi_0](t_2-t_1))\exp(iH[\phi_0(t_1)](t_1-t_2)) \\
&\quad \cdot \exp(iH_0[\phi_0](t_1-t_0))\exp(iH[\phi_0(t_0)](t_0-t_1)) \\
&= \exp(iH_0[\phi_0](t_2-t_1))\exp(iH_0[\phi_0](t_1-t_0)) \\
&\quad \cdot \exp(iH[\phi_0(t_0)](t_1-t_2))\exp(iH[\phi_0(t_0)](t_0-t_1)) \\
&= \exp(iH_0[\phi_0](t_2-t_0))\exp(iH[\phi_0(t_0)](t_0-t_2)) \\
&= U(t_2,t_1). \qquad (7.33)
\end{aligned}
$$

We write this as

$$F(x_2,x_1) = \langle 0|U(t_0,t_2)\phi_0(x_2)U(t_2,t_1)\phi_0(x_1)U(t_1,t_0)|0\rangle. \qquad (7.34)$$

Interacting Ground State. All the operators are expressed using the free field ϕ_0, but the state $|0\rangle$ is a state of the interacting theory and we do not know how to act on it.

Luckily we can express the interacting ground state $|0\rangle$ in terms of the vacuum $|0_0\rangle$ of the free theory with a trick: The free vacuum $|0_0\rangle$ should be some linear combination of the interacting ground state $|0\rangle$ and excited eigenstates $|n\rangle$ with definite energy $E_n > E_0$ [9]

$$|0_0\rangle = c_0|0\rangle + \sum_n c_n|n\rangle. \qquad (7.35)$$

Letting this state evolve for some time T with the interacting Hamiltonian we obtain

$$\exp(-iHT)|0_0\rangle = \exp(-iE_0T)c_0|0\rangle + \sum_n c_n\exp(-iE_nT)|n\rangle. \qquad (7.36)$$

All eigenstates oscillate with their respective frequencies. Suppose we give the time T some negative imaginary part with $(E_n - E_0)^{-1} \ll |\operatorname{Im}T| \ll |T|$. Then almost all eigenstates will get exponentially suppressed compared to the interacting ground state. The latter remains as the dominant contribution

$$\exp(-iHT)|0_0\rangle \approx \exp(-iE_0T)c_0|0\rangle. \qquad (7.37)$$

Primarily this identification is a formal trick. In terms of physics, we let a system in some exited state $|0_0\rangle$ evolve for some long time and find it in its ground state

---

[8]It is crucial to note in terms of which fields the respective Hamiltonians are expressed: $H_0[\phi_0(t)]$ is independent of time, while $H[\phi_0(t)]$ depends on t due to the mismatch of fields.

[9]It is reasonable to assume $c_0 = \langle 0|0_0\rangle \neq 0$ when interactions are sufficiently small.

$|0\rangle$. This is reasonable if we assume the system to be open or damped in some way. All real world systems, at least those we can expect to observe, are finite and open; the effect of $i\epsilon$ is to implement this assumption into our calculations.

We can thus express the interacting vacuum at some time t_0 as the evolution of the free vacuum at time $-T$

$$|0\rangle \simeq U(t_0, -T)|0_0\rangle, \tag{7.38}$$

where we did not pay attention to normalisation. Analogously,

$$\langle 0| \simeq \langle 0_0|U(+T, t_0). \tag{7.39}$$

Our final result for the correlation function $F(x_2, x_1)$:

$$\lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0_0|U(T, t_2)\phi_0(x_2)U(t_2, t_1)\phi_0(x_1)U(t_1, -T)|0_0\rangle}{\langle 0_0|U(T, -T)|0_0\rangle}. \tag{7.40}$$

The denominator implements the desired normalisation $\langle 0|0\rangle = 1$.

Interacting Correlators. In conclusion, the recipe for determining some correlation function in the interacting theory is the following

$$\langle 0|X|0\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0_0|U(T, t_0)XU(t_0, -T)|0_0\rangle}{\langle 0_0|U(T, -T)|0_0\rangle}. \tag{7.41}$$

where all the interacting quantum operators in X are replaced by free fields evolved from time t_0 to the desired time slice

$$\phi(t, \vec{x}) \to U(t_0, t)\phi_0(t, \vec{x})U(t, t_0). \tag{7.42}$$

Effectively two consecutive time evolution operators can always be combined into one

$$U(t_2, t_0)U(t_0, t_1) = U(t_2, t_1). \tag{7.43}$$

7.3 Perturbation Theory

We still cannot evaluate the time evolution operator $U(t, t_0)$, but at least we know that it is close to the identity when interactions are sufficiently small

$$U(t, t_0) \approx 1. \tag{7.44}$$

This approximation is too crude, it is equivalent to computing the correlator in the free theory, and we gain nothing.

Schrödinger Equation. To improve the approximation, consider the time derivative of $U(t, t_0)$

$$\begin{aligned} i\partial_t U(t, t_0) &= \exp(iH_0(t - t_0))\big(H[\phi_0(t_0)] - H_0\big) \\ &\quad \cdot \exp(-iH_0(t - t_0))U(t, t_0). \\ &= \big(H[\phi_0(t)] - H_0\big)U(t, t_0). \end{aligned} \tag{7.45}$$

We see that the time evolution operator is determined by a differential equation and a trivial initial value condition

$$i\partial_t U(t, t_0) = H_{\text{int}}(t)U(t, t_0), \qquad U(t_0, t_0) = 1. \qquad (7.46)$$

This is a Schrödinger equation, and its Hamiltonian is the so-called interaction Hamiltonian

$$H_{\text{int}}(t) := H[\phi_0(t)] - H_0[\phi_0]. \qquad (7.47)$$

This Hamiltonian is time-dependent, therefore the solution cannot be as simple as $\exp(-i(t - t_0)H_{\text{int}})$.[10] For weak interactions, one can use the Dyson series to solve the equation perturbatively.

Dyson Series. The interaction Hamiltonian is the quantity which we should assume to be small. It appears in the Schrödinger equation, so at first order we can use the above approximation for U

$$i\partial_t U(t, t_0) = H_{\text{int}}(t)U(t, t_0) \approx H_{\text{int}}(t). \qquad (7.48)$$

Integrating with proper initial value this yields

$$U(t, t_0) \approx 1 - i \int_{t_0}^{t} dt_1 H_{\text{int}}(t_1). \qquad (7.49)$$

This is certainly better than before, it involves interactions at first order. Nevertheless we can do better.

To go further systematically, write the differential equation in integral form

$$U(t, t_0) = 1 - i \int_{t_0}^{t} dt_1 H_{\text{int}}(t_1)U(t_1, t_0). \qquad (7.50)$$

Substitute the above solution yields a better solution

$$U(t, t_0) \approx 1 - i \int_{t_0}^{t} dt_1 H_{\text{int}}(t_1)$$
$$- \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_{\text{int}}(t_1) H_{\text{int}}(t_2). \qquad (7.51)$$

Now use the new solution instead

$$U(t, t_0) \approx 1 - i \int_{t_0}^{t} dt_1 H_{\text{int}}(t_1)$$
$$- \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_{\text{int}}(t_1) H_{\text{int}}(t_2)$$
$$+ i \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 H_{\text{int}}(t_1) H_{\text{int}}(t_2) H_{\text{int}}(t_3). \qquad (7.52)$$

And so on.

The picture should be clear, we could go to arbitrarily high orders. More importantly, everything is expressed in terms of free fields ϕ_0 and the interaction Hamiltonian $H_{\text{int}}[\phi_0]$.

---

[10]H_{int} is time-dependent because its time evolution is governed by H_0 with which it does not commute in general.

Time-Ordered Exponential.   The multiple integral with a nested sequence of boundaries is hard to handle. We can improve the situation. Consider the quadratic term:

$$-\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_{\mathrm{int}}(t_1)\, H_{\mathrm{int}}(t_2). \tag{7.53}$$

We can also write it as

$$-\int_{t_0}^{t} dt_1 \int_{t_1}^{t_2} dt_2\, H_{\mathrm{int}}(t_2)\, H_{\mathrm{int}}(t_1). \tag{7.54}$$

The integration region assumes $t_1 \geq t_2$ in the first integral and $t_2 \geq t_1$ in the second integral. Importantly, in both integrands the operator $H_{\mathrm{int}}(t_k)$ with larger t_k is to the right of the operator $H_{\mathrm{int}}(t_j)$ with smaller t_j.

$$(7.55)$$
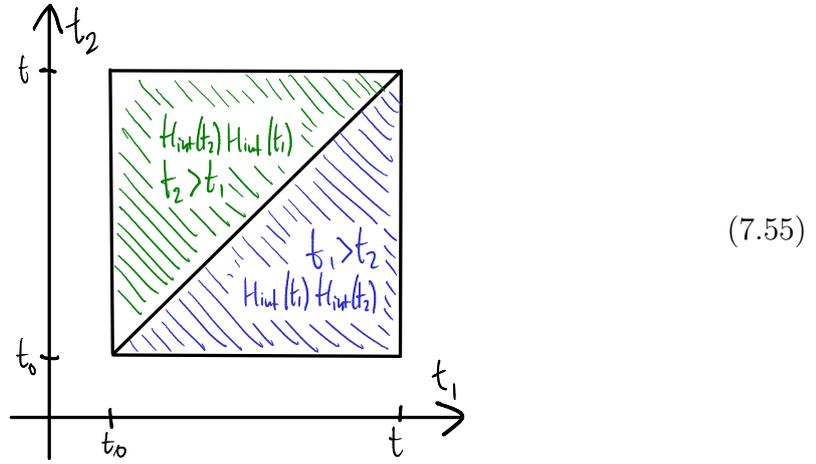
We introduce a time ordering symbol T which puts the affected operators in an order with time decreasing from left to right, e.g.[11]

$$\mathrm{T}\big(X(t_1)Y(t_2)\big) := \begin{cases} X(t_1)Y(t_2) & \text{for } t_1 > t_2, \\ Y(t_2)X(t_1) & \text{for } t_1 < t_2, \end{cases} \tag{7.56}$$

and similarly for multiple operators. This allows to write the integrand of both above integrals as $\mathrm{T}(H_{\mathrm{int}}(t_1)\, H_{\mathrm{int}}(t_2))$. We can thus write the integral as the average of the two equivalent representations where the integration regions combine to a square

$$-\frac{1}{2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\, \mathrm{T}\big(H_{\mathrm{int}}(t_1)\, H_{\mathrm{int}}(t_2)\big). \tag{7.57}$$

Even better, we can write this as the time-ordered square of a single integral

$$-\frac{1}{2}\mathrm{T}\left(\int_{t_0}^{t} dt'\, H_{\mathrm{int}}(t')\right)^2 \tag{7.58}$$

---

[11]For fermionic operators X, Y one would insert suitable signs for flipping the order.

As all terms of the perturbative expansion of $U(t, t_0)$ are naturally in time ordering, the above construction generalises straight-forwardly to the n-th order term

$$\frac{1}{n!} \mathrm{T} \left(-i \int_{t_0}^{t} dt'\, H_{\mathrm{int}}(t') \right)^n.$$

(7.59)

Here the integration region is a hypercube in n dimensions. It contains $n!$ simplices,[12] which form the integration regions for the terms in the Dyson series.

Summing up all terms yields the time-ordered exponential

$$U(t_2, t_1) = \mathrm{T} \exp\left(i S_{\mathrm{int}}(t_1, t_2) \right).$$

(7.60)

where we introduced the interaction action S_{int} between times t_1 and t_2

$$S_{\mathrm{int}}(t_1, t_2) := -\int_{t_1}^{t_2} dt'\, H_{\mathrm{int}}(t').$$

(7.61)

The time-ordered exponential represents both the formal solution to the above Schrödinger equation for $U(t_2, t_1)$ as well as a concrete perturbative prescription to evaluate it.

---

[12] A hypercube is the generalisation of a cube to n dimensions, a simplex is the generalisation of a triangle.

8 Correlation Functions

We have seen how to formally write the time evolution operator

$$U(t_1, t_0) = \mathrm{T} \exp\left(i S_{\mathrm{int}}(t_0, t_1)\right) \tag{8.1}$$

in an interacting QFT model based on the interaction picture and time-ordered products.

A particularly convenient correlator is one where the operators are already in proper time order

$$\langle X[\phi] \rangle := \langle 0 | \mathrm{T}\left(X[\phi]\right) | 0 \rangle. \tag{8.2}$$

Such time-ordered correlation functions have multiple applications in QFT, for example, it can be used for particle scattering processed. In this chapter we will develop methods to compute them in more practical terms. The outcome will be a set graphical rules, the Feynman rules.

For simplicity we will drop all free field indices $\phi_0 \to \phi$ from now on and instead mark interacting correlators by an index "int".

8.1 Interacting Time-Ordered Correlators

Consider the correlator of two time-ordered fields with $t_1 > t_2$

$$F = \langle \phi(t_1, \vec{x}_1) \phi(t_2, \vec{x}_2) \rangle_{\mathrm{int}} = \langle 0_{\mathrm{int}} | \phi_{\mathrm{int}}(t_1, \vec{x}_1) \phi_{\mathrm{int}}(t_2, \vec{x}_2) | 0_{\mathrm{int}} \rangle. \tag{8.3}$$

In the expression in terms of free fields

$$X = U(T, t_2) \phi(x_2) U(t_2, t_1) \phi(x_1) U(t_1, -T), \tag{8.4}$$

we notice that all operators are in proper time order and we can extend the time ordering over all the operators

$$
\begin{aligned}
X ={}& \left(\mathrm{T}\exp\left(i S_{\mathrm{int}}(t_2, T)\right)\right) \phi(x_2) \left(\mathrm{T}\exp\left(i S_{\mathrm{int}}(t_1, t_2)\right)\right) \\
& \cdot \phi(x_1) \left(\mathrm{T}\exp\left(i S_{\mathrm{int}}(-T, t_1)\right)\right) \\
={}& \mathrm{T}\big(\exp\left(i S_{\mathrm{int}}(t_2, T)\right) \phi(x_2) \exp\left(i S_{\mathrm{int}}(t_1, t_2)\right) \\
& \cdot \phi(x_1) \exp\left(i S_{\mathrm{int}}(-T, t_1)\right)\big).
\end{aligned}
\tag{8.5}
$$

Inside the time-ordering symbol the order of operators does not matter. The exponents can now be combined nicely:

$$X = \mathrm{T}\big(\phi(x_1)\phi(x_2) \exp\left(i S_{\mathrm{int}}(-T, T)\right)\big). \tag{8.6}$$

We thus find the correlation function $\langle \phi(x_1)\phi(x_2) \rangle_{\mathrm{int}}$

$$F = \lim_{T \to \infty(1 - i\epsilon)} \frac{\langle 0 | \mathrm{T}\big(\phi(x_1)\phi(x_2) \exp(i S_{\mathrm{int}}(-T, T))\big) | 0 \rangle}{\langle 0 | \mathrm{T}\big(\exp(i S_{\mathrm{int}}(-T, T))\big) | 0 \rangle}. \tag{8.7}$$

This formula generalises to vacuum expectation values of arbitrary time-ordered combinations $X[\phi]$ of quantum operators

$$\langle \mathrm{T}(X[\phi])\rangle_{\mathrm{int}} = \lim_{T \to \infty(1-i\epsilon)} \frac{\langle 0|\mathrm{T}\big(X[\phi]\exp(iS_{\mathrm{int}}(-T,T))\big)|0\rangle}{\langle 0|\mathrm{T}\big(\exp(iS_{\mathrm{int}}(-T,T))\big)|0\rangle}$$
$$\simeq \frac{\langle 0|\mathrm{T}\big(X[\phi]\exp(iS_{\mathrm{int}})\big)|0\rangle}{\langle 0|\mathrm{T}\big(\exp(iS_{\mathrm{int}})\big)|0\rangle}. \tag{8.8}$$

Here the complete interaction action S_{int} implies a small imaginary part for the time coordinate in the distant past and future. We can thus express time-ordered correlators in the interacting theory in terms of similar quantities in the free theory.

This expression has several benefits and applications:

- Typically there are no ordering issues within X because time ordering puts all constituent operators into some well-defined order. This is useful when interested in the quantum expectation value of some product of classical operators.
- It directly uses the interaction terms S_{int} in the action.
- Time-ordered products and expectation values can be evaluated conveniently.
- This expression appears in many useful observables, for example in particle scattering amplitudes.

8.2   Time-Ordered Products

We now look for a method to evaluate a time-ordered correlator of a combination of free field operators $X[\phi]$

$$\langle X[\phi]\rangle := \langle 0|\mathrm{T}\big(X[\phi]\big)|0\rangle. \tag{8.9}$$

Feynman Propagator.   We start with two fields

$$G_{\mathrm{F}}(x_1, x_2) = i\langle 0|\mathrm{T}\big(\phi(x_1)\phi(x_2)\big)|0\rangle. \tag{8.10}$$

By construction and earlier results it reads

$$G_{\mathrm{F}}(x_1, x_2) = \begin{cases} i\langle 0|\phi(x_1)\phi(x_2)|0\rangle & \text{for } t_1 > t_2, \\ i\langle 0|\phi(x_2)\phi(x_1)|0\rangle & \text{for } t_2 > t_1 \end{cases}$$
$$= i\theta(t_1 - t_2)\Delta_+(x_1 - x_2) + i\theta(t_2 - t_1)\Delta_+(x_2 - x_1). \tag{8.11}$$

Comparing this to the retarded propagator $G_{\mathrm{R}}(x)$

$$G_{\mathrm{F}}(x) = i\theta(t)\Delta_+(x) + i\theta(-t)\Delta_+(-x),$$
$$G_{\mathrm{R}}(x) = i\theta(t)\Delta_+(x) - i\theta(t)\Delta_+(-x), \tag{8.12}$$

we can write

$$G_{\mathrm{R}}(x) = G_{\mathrm{F}}(x) - i\Delta_+(-x). \tag{8.13}$$

As such it obeys the equation of a propagator,

$$-\partial^2 G_{\mathrm{F}}(x) + m^2 G_{\mathrm{F}}(x) = \delta^{d+1}(x), \tag{8.14}$$

but with different boundary conditions than of the retarded propagator. It is called the Feynman propagator.

The momentum space representation of the Feynman propagator for the scalar field reads

$$G_{\mathrm{F}}(p) = \frac{1}{p^2 + m^2 - i\epsilon}. \tag{8.15}$$

Here the two poles at $e = \pm e(\vec{p})$ are shifted up and down into the complex plane by a tiny amount

$$G_{\mathrm{F}}(p) = \frac{1}{2e(\vec{p})} \left(\frac{1}{e - (-e(\vec{p}) + i\epsilon)} - \frac{1}{e - (+e(\vec{p}) - i\epsilon)} \right). \tag{8.16}$$

$$\tag{8.17}$$
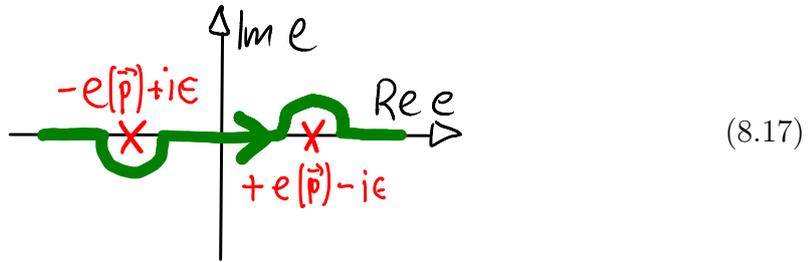
Concerning the relation to the position space representation:

- The positive energy pole $e = e(\vec{p}) - i\epsilon$ is below the real axis and thus relevant to positive times.
- The negative energy pole $e = -e(\vec{p}) + i\epsilon$ is above the real axis and thus relevant to negative times.

Alternatively, to obtain the correct contour around the two poles, we could integrate on a slightly tilted energy axis in the complex plane

$$e \sim (1 + i\epsilon). \tag{8.18}$$

Note that this corresponds to assuming times to be slightly imaginary, but in the opposite direction such that et is real

$$t \sim (1 - i\epsilon). \tag{8.19}$$

The $i\epsilon$ prescription of the Feynman propagators is thus directly related and equivalent to the $i\epsilon$ prescription for converting the free vacuum to the interacting one.

$$\tag{8.20}$$

8.3
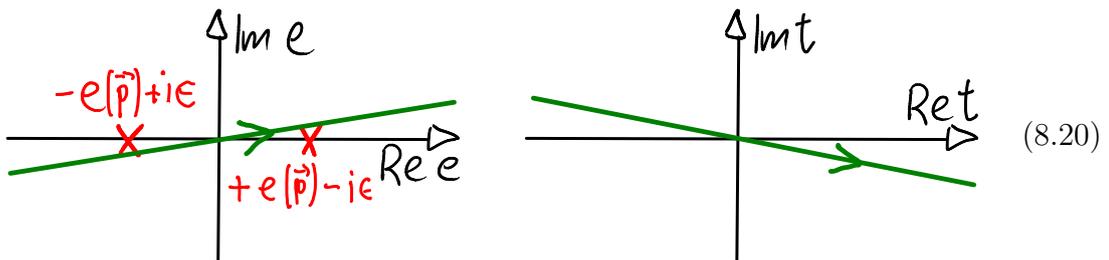
Wick's Theorem. To evaluate more complex time-ordered vacuum expectation values one typically employs Wick's theorem. It relates a time-ordered product of operators $\mathrm{T}(X[\phi])$ to a normal-ordered product of operators $\mathrm{N}(X[\phi])$. The normal-ordered product is useful when evaluating vacuum expectation values because the VEV picks out field-independent contributions only.

Let us recall the definition of normal ordering: Split up the free fields ϕ into pure creation operators ϕ^+ and pure annihilation operators ϕ^-

$$\phi = \phi^+ + \phi^-, \qquad \phi^+ \sim a^\dagger, \quad \phi^- \sim a. \tag{8.21}$$

Normal ordering of a product is defined such that all factors of ϕ^+ are to the left of all factors ϕ^-. For example,

$$\begin{aligned}
\mathrm{N}\big(\phi(x_1)\phi(x_2)\big) &= \phi^+(x_1)\phi^+(x_2) + \phi^-(x_1)\phi^-(x_2) \\
&\quad + \phi^+(x_1)\phi^-(x_2) + \phi^+(x_2)\phi^-(x_1),
\end{aligned} \tag{8.22}$$

where the latter two terms are in normal order and the ordering of the former two terms is irrelevant.

In comparison, time-ordering of the same product is defined as

$$\begin{aligned}
\mathrm{T}\big(\phi(x_1)\phi(x_2)\big) &= \phi^+(x_1)\phi^+(x_2) + \phi^-(x_1)\phi^-(x_2) \\
&\quad + \theta(t_2 - t_1)\big(\phi^+(x_2)\phi^-(x_1) + \phi^-(x_2)\phi^+(x_1)\big) \\
&\quad + \theta(t_1 - t_2)\big(\phi^+(x_1)\phi^-(x_2) + \phi^-(x_1)\phi^+(x_2)\big).
\end{aligned} \tag{8.23}$$

The difference between the two expressions reads

$$\begin{aligned}
(\mathrm{T} - \mathrm{N})\big(\phi(x_1)\phi(x_2)\big) &= \theta(t_2 - t_1)[\phi^-(x_2), \phi^+(x_1)] \\
&\quad + \theta(t_1 - t_2)[\phi^-(x_1), \phi^+(x_2)] \\
&= \theta(t_2 - t_1)\Delta_+(x_2 - x_1) \\
&\quad + \theta(t_1 - t_2)\Delta_+(x_1 - x_2) \\
&= -iG_F(x_1 - x_2).
\end{aligned} \tag{8.24}$$

Wick's theorem is a generalisation of this result to an arbitrary number of fields: It states that the time-ordered product of a set of fields equals the partially contracted normal-ordered products summed over multiple contractions between pairs of fields. A Wick contraction between two, not necessarily adjacent, fields $\phi(x_k)$ and $\phi(x_l)$ replaces the relevant two field operators by their Feynman propagator $-iG_\mathrm{F}(x_k, x_l)$, in short:

$$\begin{aligned}
&[\ldots \phi_{k-1}\underbrace{\phi_k\phi_{k+1}\ldots\phi_{l-1}\phi_l}\phi_{l+1}\ldots] \\
&:= -iG_\mathrm{F}(x_1 - x_2)[\ldots\phi_{k-1}\ \phi_{k+1}\ldots\phi_{l-1}\ \phi_{l+1}\ldots].
\end{aligned} \tag{8.25}$$

For example:

$$T\big(\phi_1\phi_2\big) = N\big(\phi_1\phi_2\big) + \contraction{}{\phi_1}{}{\phi_2}\phi_1\phi_2,$$

$$T\big(\phi_1\phi_2\phi_3\big) = N\big(\phi_1\phi_2\phi_3\big) + \phi_1\phi_2\phi_3 + \phi_1\phi_2\phi_3 + \phi_1\phi_2\phi_3,$$

$$T\big(\phi_1\phi_2\phi_3\phi_4\big) = N\Big[\phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4$$
$$+ \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4$$
$$+ \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4\Big]$$
$$+ \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4, \tag{8.26}$$

To prove the statement by induction is straight-forward:

- Assume the statement holds for $n-1$ fields.
- Arrange n fields in proper time order $\phi_n\dots\phi_1$ with $t_n > \dots > t_1$.
- Consider $T[\phi_n\dots\phi_1] = (\phi_n^+ + \phi_n^-)T[\phi_{n-1}\dots\phi_1]$ and replace $T[\phi_{n-1}\dots\phi_1]$ by contracted normal-ordered products.
- ϕ_n^+ is already in normal order, it can be pulled into $N[\dots]$.
- Commute ϕ_n^- past all the remaining fields in the normal ordering.
- For every uncontracted field ϕ_k in $N[\dots]$, pick up a term
  $\Delta_+(x_n - x_k) = -iG_F(x_n - x_k)$ because $t_n > t_k$.
- Convince yourself that all contractions of n fields are realised with unit weight.
- Convince yourself that for different original time-orderings of $\phi_n\dots\phi_1$, the step functions in G_F do their proper job.

Time-Ordered Correlators. To compute time-ordered correlators we can use the result of Wick's theorem. All the normal-ordered terms with remaining fields drop out of vacuum expectation values. The only terms to survive are those where all the fields are complete contracted in pairs

$$\langle\phi_1\dots\phi_n\rangle := \sum_{\substack{\text{complete}\\\text{contractions}}} \phi_1\phi_2\dots\dots\dots\dots\phi_{n-1}\phi_n. \tag{8.27}$$

In particular, it implies that correlators of an odd number of fields must be zero.

This formula applies directly to a single species of real scalar fields, but for all the other fields and mixed products there are a straight-forward equivalents:

- For fields with spin, use the appropriate propagator, e.g. $(G^D)^a{}_b$ for contracting the Dirac fields ψ^a and $\bar\psi_b$.
- For any crossing of lines attached to fermionic fields, multiply by a factor of (-1).

8.3 Some Examples

We have learned how to reduce time-ordered correlators in a weakly interacting QFT to free time-ordered correlators

$$\langle X[\phi] \rangle_{\text{int}} = \frac{\langle X[\phi] \, \exp(iS_{\text{int}}[\phi]) \rangle}{\langle \exp(iS_{\text{int}}[\phi]) \rangle} . \tag{8.28}$$

We have also learned how to evaluate the latter

$$\langle \phi_1 \ldots \phi_n \rangle := \sum_{\substack{\text{complete} \\ \text{contractions}}} \phi_1 \phi_2 \ldots\ldots\ldots\ldots \phi_{n-1} \phi_n . \tag{8.29}$$

We will now apply these formulas to some basic types of time-ordered correlators in order to develop an understanding for them.

Setup. We will consider ϕ^4 theory, i.e. a single real scalar field with a ϕ^4 interaction

$$\mathcal{L} = -\tfrac{1}{2} \partial^\mu \phi \, \partial_\mu \phi - \tfrac{1}{2} m^2 \phi^2 - \tfrac{1}{24} \lambda \phi^4 . \tag{8.30}$$

We define the interaction picture using the quadratic terms in the action

$$\mathcal{L}_0 = -\tfrac{1}{2} \partial^\mu \phi \, \partial_\mu \phi - \tfrac{1}{2} m^2 \phi^2 . \tag{8.31}$$

What remains is the interaction term

$$\mathcal{L}_{\text{int}} = -\tfrac{1}{24} \lambda \phi^4 , \tag{8.32}$$

whose coefficient, the coupling constant λ, is assumed to be small. The interaction part of the action is thus

$$S_{\text{int}}(t_1, t_2) := \int_{t_1}^{t_2} dt \int d^3\vec{x} \, \mathcal{L}_{\text{int}}(x), \quad S_{\text{int}} := S_{\text{int}}(-\infty, +\infty). \tag{8.33}$$

We would like to evaluate the correlators of two and four fields

$$T_{12} = \langle \phi_1 \phi_2 \rangle_{\text{int}}, \qquad F_{1234} = \langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle_{\text{int}}, \tag{8.34}$$

where ϕ_k denotes the field $\phi(x_k)$ evaluated at position x_k. These are functions of the coupling constant λ which we formally expand for small λ as

$$T(\lambda) = \sum_{n=0}^{\infty} T^{(n)}, \quad F(\lambda) = \sum_{n=0}^{\infty} F^{(n)}, \qquad T^{(n)} \sim F^{(n)} \sim \lambda^n . \tag{8.35}$$

Leading Order. First, we shall evaluate T and F at lowest order in the coupling strength. At leading order we simply set $\lambda = 0$ and obtain the correlator in the free theory

$$T^{(0)} = \langle \phi_1 \phi_2 \rangle, \qquad F^{(0)} = \langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle. \tag{8.36}$$

Using Wick's theorem this evaluates to

$$T^{(0)} = \underbracket{\phi_1 \phi_2},$$
$$F^{(0)} = \underbracket{\phi_1 \phi_2}\,\underbracket{\phi_3 \phi_4} + \underbracket{\phi_1 \phi_2 \phi_3 \phi_4} + \underbracket{\phi_1 \phi_2 \phi_3 \phi_4}. \tag{8.37}$$

More formally, these equal

$$T^{(0)} = (-i) G_{12},$$
$$F^{(0)} = (-i)^2 G_{12} G_{34} + (-i)^2 G_{13} G_{24} + (-i)^2 G_{14} G_{23}, \tag{8.38}$$

where G_{kl} denotes $G_{\mathrm{F}}(x_k - x_l)$. In a graphical notation we could write this as

$$T^{(0)} = \quad\text{(diagram)}\quad,$$

$$F^{(0)} = \quad\text{(diagrams)}\quad. \tag{8.39}$$

Each vertex represents a spacetime point x_k and each line connecting two vertices k and l represents a propagator $-i G_{\mathrm{F}}(x_k - x_l)$.

Two-Point Function at First Order. The contributions to the interacting two-point function at the next perturbative order read

$$T^{(1)} = \langle \phi_1 \phi_2 i S_{\mathrm{int}}[\phi] \rangle - \langle \phi_1 \phi_2 \rangle \langle i S_{\mathrm{int}}[\phi] \rangle$$
$$= -\frac{i\lambda}{24} \int d^4 y \, \langle \phi_1 \phi_2 \phi_y \phi_y \phi_y \phi_y \rangle$$
$$+ \frac{i\lambda}{24} \int d^4 y \, \langle \phi_1 \phi_2 \rangle \langle \phi_y \phi_y \phi_y \phi_y \rangle. \tag{8.40}$$

Using Wick's theorem the two terms expand to 15 and 3 contributions. Consider the first term only: The 15 contributions can be grouped into two types. The first type receives 12 identical contributions from contracting the 4 identical ϕ_y's in superficially different ways. The remaining 3 terms in the second group are identical for the same reason. We summarise the groups as follows

$$T^{(1)}_{\mathrm{1a}} = -\tfrac{i}{2}\lambda \int d^4 y \, \underbracket{\phi_1 \phi_2 \phi_y \phi_y \phi_y \phi_y} = -\tfrac{i}{2}\lambda \int d^4 y \quad\text{(diagram)},$$

$$T^{(1)}_{\mathrm{1b}} = -\tfrac{i}{8}\lambda \int d^4 y \, \underbracket{\phi_1 \phi_2 \phi_y \phi_y \phi_y \phi_y} = -\tfrac{i}{8}\lambda \int d^4 y \quad\text{(diagram)}. \tag{8.41}$$

8.7

The term originating from the denominator of the interacting correlator evaluates to

$$T_2^{(1)} = -\tfrac{1}{8}(-i\lambda) \int d^4y\, \phi_1\phi_2\phi_y\phi_y\phi_y\phi_y = -T_{1b}^{(1)}.$$
(8.42)

It precisely cancels the second contribution to the first term. Altogether we find the following expression for the first-order correction to the two-point function

$$T^{(1)} = T_{1a}^{(1)} = \tfrac{1}{2}(-i)^4\lambda \int d^4y\, G_{1y}G_{2y}G_{yy}$$

$$= -\tfrac{i}{2}\lambda \int d^4y$$
(8.43)

Tadpoles. We were careful enough not to write this expression too explicitly

$$T^{(1)} = \tfrac{1}{2}\lambda\, G_{\mathrm{F}}(0) \int d^4y\, G_{\mathrm{F}}(x_1 - y)G_{\mathrm{F}}(x_2 - y).$$
(8.44)

We notice that one of the propagators decouples from the function. Moreover its argument is precisely zero because the propagator connects a point to itself.

- The result is in general divergent, it is very similar to the vacuum energy we encountered much earlier in QFT.
- In our derivation of time ordering we were sloppy in that we did not discuss the case of equal times. In a local Lagrangian, however, all terms are defined at equal time, moreover at equal spatial position. It would make sense to employ normal ordering in this case, which eliminates the term at the start.
- Whatever the numerical value of $G_{\mathrm{F}}(0)$, even if infinite, it does not yield any interesting functional dependence to $T^{(1)}$. In fact it could be eliminated by adding a term $-\tfrac{i}{4}\lambda G_{\mathrm{F}}(0)\phi^2$ to the interaction Lagrangian. This has the same effect as normal ordering the Lagrangian.

This term is called a tadpole term because the corresponding diagram looks like a tadpole sitting on the propagator line

(8.45)
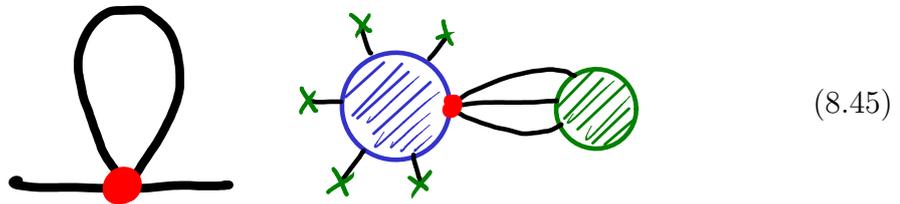
More generally, tadpoles are internal parts of a diagram which are attached to the rest of the diagram only via a single vertex. In most cases, they can be compensated by adding suitable local terms to the interaction Lagrangian. Even though this correction term is somewhat dangerous and somewhat trivial, let us pretend it is a regular contribution and carry it along.

First Order Four-Point Function. The first-order contributions to the interacting four-point function take a similar form

$$F^{(1)} = \langle \phi_1 \phi_2 \phi_3 \phi_4 i S_{\text{int}}[\phi] \rangle - \langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle \langle i S_{\text{int}}[\phi] \rangle$$

$$= -\frac{i\lambda}{24} \int d^4y \, \langle \phi_1 \phi_2 \phi_3 \phi_4 \phi_y \phi_y \phi_y \phi_y \rangle$$

$$+ \frac{i\lambda}{24} \int d^4y \, \langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle \langle \phi_y \phi_y \phi_y \phi_y \rangle. \tag{8.46}$$

These expressions are not as innocent as they may look: Using Wick's theorem the two terms expand to $7 \cdot 5 \cdot 3 \cdot 1 = 105$ and $3 \cdot 3 = 9$ terms. Gladly, most of these terms are identical and can be summarised, we group them into 24, $6 \cdot 12$ and $3 \cdot 3$ terms from the first contribution and $3 \cdot 3$ terms from the second one

$$F_{1a}^{(1)} = (-i\lambda) \int d^4y \, \phi_1 \phi_2 \phi_3 \phi_4 \phi_y \phi_y \phi_y \phi_y,$$

$$F_{1b}^{(1)} = \tfrac{1}{2}(-i\lambda) \int d^4y \, \phi_1 \phi_2 \phi_3 \phi_4 \phi_y \phi_y \phi_y \phi_y + 5 \text{ perm.},$$

$$F_{1c}^{(1)} = \tfrac{1}{8}(-i\lambda) \int d^4y \, \phi_1 \phi_2 \phi_3 \phi_4 \phi_y \phi_y \phi_y \phi_y + 2 \text{ perm.},$$

$$F_2^{(1)} = -\tfrac{1}{8}(-i\lambda) \int d^4y \, \phi_1 \phi_2 \phi_3 \phi_4 \phi_y \phi_y \phi_y \phi_y + 2 \text{ perm.}. \tag{8.47}$$

The graphical representation of these terms is

$$\tag{8.48}$$

Let us now discuss the roles of the three terms.

Vacuum Bubbles. In the above result we notice that again the contribution $F_2^{(1)}$ from the denominator of the interacting correlator cancels the term $F_{1c}^{(1)}$ from the numerator. This effect is general:

Some graphs have components which are coupled neither to the rest of the graph nor to the external points. Such parts of the graph are called vacuum bubbles.

- Vacuum bubbles represent some virtual particles which pop out of the quantum mechanical vacuum and annihilate among themselves. They do not interact with any of the physically observed particles, hence one should be able to ignore such contributions.
- Vacuum bubbles are usually infinite. Here we obtain as coefficient $G_F(0)^2 \int d^4y$. This contains two divergent factors of $G_F(0)$ and an infinite spacetime volume $\int d^4y$.
- Formally, we could remove such terms by adding a suitable field-independent term to the Lagrangian. Alternatively we could normal order it.
- In any case, vacuum bubbles are generally removed by the denominator of the interacting correlation function. This cancellation ensures that the interacting vacuum is properly normalised, $\langle 0|0 \rangle_{\text{int}} = 1$. Any diagram containing at least one vacuum bubble can be discarded right away.

Disconnected Graphs. The contribution from $F_{1b}^{(1)}$ is reminiscent of the correction $T^{(1)}$ to the two-point function. In fact it can be written as a sum of products of two-point functions

$$
\begin{aligned}
F_{1b}^{(1)} = {}& T_{12}^{(0)}T_{34}^{(1)} + T_{13}^{(0)}T_{24}^{(1)} + T_{14}^{(0)}T_{23}^{(1)} \\
& + T_{12}^{(1)}T_{34}^{(0)} + T_{13}^{(1)}T_{24}^{(0)} + T_{14}^{(1)}T_{23}^{(0)}.
\end{aligned}
\tag{8.49}
$$

This combination is precisely the first-order contribution to a product of two $T(\lambda)$'s
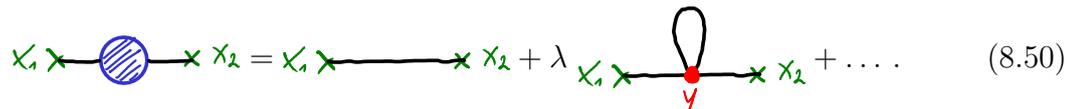
$$
\tag{8.50}
$$

This is also a general feature of correlation functions:

- Correlation functions contain disconnected products of lower-point functions. The corresponding graphs contain disconnected components (each of which is connected to at least one external field).
- Such contributions are typically put aside because their form is predictable.[1] Nevertheless, they are essential and non-negligible contributions.
- Such disconnected contributions represent processes that take place simultaneously without interfering with each other.

---

[1]When computing an n-point function one will typically already have computed all the k-point functions with $k < n$ anyway.

Quite generally one can split the contributions into connected and disconnected terms. Here we know, to all orders in λ

$$F(\lambda) = T_{12}(\lambda)T_{34}(\lambda) + T_{13}(\lambda)T_{24}(\lambda) + T_{14}(\lambda)T_{23}(\lambda)$$
$$+ F_{\text{conn}}(\lambda)$$

$$(8.51)$$

where $F_{\text{conn}}(\lambda)$ summarises all connected contributions. In our case

$$F_{\text{conn}}^{(1)} = F_{1a}^{(1)} = -i\lambda \int d^4y \, G_{1y}G_{2y}G_{3y}G_{4y}$$

$$(8.52)$$

Symmetry Factors. In our computation, we have encountered many equivalent contributions which summed up into a single term. We observe that these sums have conspired to cancel most of the prefactors of $1/24$. The purpose of having a prefactor of $1/24$ for ϕ^4 in the action is precisely to be cancelled against multiplicities in correlators, where λ typically appears without or with small denominators.

We can avoid constructing a large number of copies of the same term by considering the symmetry of terms or the corresponding graphs. The symmetry factor is the inverse size of the discrete group that permutes the elements of a term or a graph while leaving its structure invariant.

To make use of symmetry factors for the calculation of correlation functions, one should set up the Lagrangian such that every product of terms comes with the appropriate symmetry factor. For example, the term ϕ^4 allows arbitrary permutations of the 4 ϕ's. There are $4! = 24$ such permutations, hence the appropriate symmetry factor is $1/24$.[2]

The crucial insight is the following: When the symmetry factors for the Lagrangian are set up properly, the summed contributions to correlation function also have their appropriate symmetry factors.

---

[2]After all, we are free to call the term that multiplies ϕ^4 either $\lambda/24$ or λ'. It is not even difficult to translate between them.

To determine the symmetry factors correctly sometimes is difficult, as one has to identify all permissible permutations. This can be difficult, for example, when the graphical representation has a difficult topology or when it hides some relevant information.

Let us consider the symmetry factors of the terms we have computed so far. The contributions $T^{(0)}$, $F^{(0)}$ and $F_{\text{conn}}^{(1)}$ have trivial symmetry factors.

$$(8.53)$$

Permutations of any of the elements would change the labelling of the external fields. The symmetry factor for the tadpole diagram is $1/2$.

$$(8.54)$$

The relevant $\mathbb{Z}_2$ symmetry flips the direction of the tadpole line.[3] Finally, the vacuum bubble diagram has a symmetry factor of $1/8$.

$$(8.55)$$

There are two factors of $1/2$ for flipping the direction of the tadpole lines. Then there is another factor of $1/2$ for permuting the two tadpole lines.

8.4 Feynman Rules

We have seen how to evaluate some perturbative contributions to interacting correlators. Following the formal prescription leads to a lot of combinatorial overhead as the results tend to be reasonably simple compared to the necessary intermediate steps. Feynman turned the logic around and proposed a simple graphical construction of correlators:

The interacting correlator of several fields can be expressed as a sum of so-called Feynman graphs. Each Feynman graph represents a certain mathematical expression which can be evaluated from the graph by the Feynman rules. Moreover a Feynman graph display nicely the physical process that leads to the corresponding term of the correlator.

---

[3] In fact, the symmetry acts on the connections of lines to vertices. Here, exchanging the two endpoints of the tadpole line is the only symmetry.

For every weakly coupled QFT there is a set of Feynman rules to compute its correlators.[4] Here we list the Feynman rules for the scalar ϕ^4 model.

Feynman Rules in Position Space. A permissible graph for a correlator

$$F(x_1, \ldots, x_n) = \langle \phi(x_1) \ldots \phi(x_n) \rangle_{\text{int}} \tag{8.56}$$

- has undirected and unlabelled edges,

$$\tag{8.57}$$

- has n 1-valent (external) vertices labelled by x_j,

$$\tag{8.58}$$

- has an arbitrary number k of 4-valent (internal) vertices labelled by y_j,
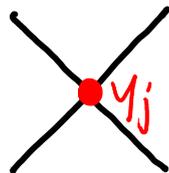
$$\tag{8.59}$$

- can have lines connecting a vertex to itself (tadpole),
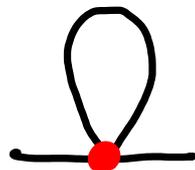
$$\tag{8.60}$$

- can have several connection components,
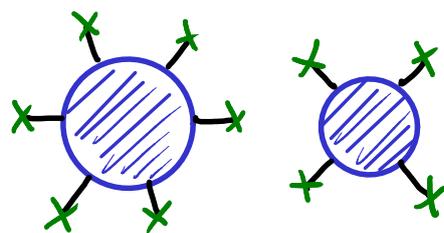
$$\tag{8.61}$$

- must not have components disconnected from all of the external vertices x_j (vacuum bubble),
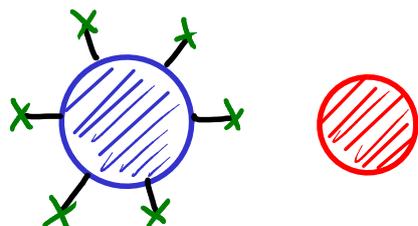
$$\tag{8.62}$$

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[4]Similar graphs and rules can actually be set up and applied to a wide range of algebraic problems not at all limited to relativistic QFT's.

For each topologically distinct graph we can compute a contribution according to the following rules:

- For each edge connecting two vertices z_i and z_j write a factor of $-iG_{\mathrm{F}}(z_i - z_j)$.

$$z_i \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet z_j \to -iG_{\mathrm{F}}(z_i - z_j) \tag{8.63}$$

- For each 4-valent vertex y_j, write a factor of $-i\lambda$ and integrate over $\int d^4 y_j$.

$$y_j \to -i\lambda \int d^4 y_j \tag{8.64}$$

- Multiply by the appropriate symmetry factor, i.e. divide by the number of discrete symmetries of the graph.

Feynman Rules in Momentum Space. One of the problems we have not yet mentioned is that the Feynman propagator G_{F} is a complicated function in spacetime. Moreover, we need to compute multiple convolution integrals of these functions over spacetime, e.g. the integral defining $F^{(1)}_{\mathrm{conn}}$. This soon enough exceeds our capabilities.

These computations can be simplified to some extent by going to momentum space. Such a momentum space representation will be particularly useful later when we compute the interaction between particles with definite momenta in particle scattering experiments.

The momentum space version is defined as follows[5]

$$F(p_1, \ldots, p_n) = \int d^4 x_1 \ldots d^4 x_n \, e^{ix_1 \cdot p_1 + \ldots + ix_n \cdot p_n} \langle \phi_1 \ldots \phi_n \rangle. \tag{8.65}$$

A Feynman graph in momentum space

- has edges labelled by a directed flow of 4-momentum ℓ_j from one end to the other,

$$\ell_j \to \tag{8.66}$$

- has n 1-valent (external) vertices with an inflow of 4-momentum p_j,

$$p_j \tag{8.67}$$

- has an arbitrary number k of 4-valent (internal) vertices which conserve the flow of momentum
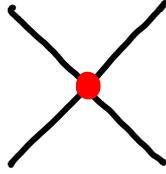
(8.68)

- shares the remaining attributes with the position space version.

The Feynman rules for evaluating a graph read

- Work out the flow of momentum from the external vertices across the internal vertices. Label all edges with the appropriate momenta ℓ_j.
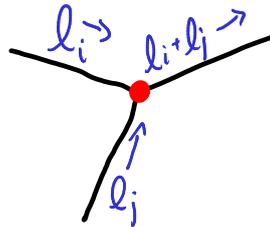
(8.69)

- There is a momentum-conservation condition $p_{j_1} + \ldots + p_{j_m}$ for each connected component of the graph. Write a factor of $(2\pi)^4 \delta^4(p_{j_1} + \ldots + p_{j_m})$ including all contributing external momenta p_j.
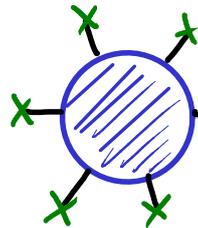
$$\to (2\pi)^4 \delta^4(p_{j_1} + \ldots + p_{j_m})$$

(8.70)

- For each internal loop of the graph, there is one undetermined 4-momentum ℓ_j. Integrate the final expression over all such momenta $\int d^4\ell_j/(2\pi)^4$.
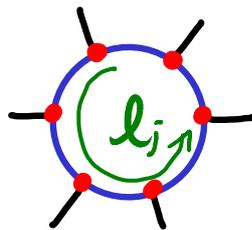
$$\to \int \frac{d^4\ell_j}{(2\pi)^4}$$

(8.71)

- For each edge write a factor of $-i/(\ell_j^2 + m^2 - i\epsilon)$.
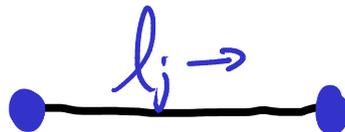
$$\to \frac{-i}{\ell_j^2 + m^2 - i\epsilon}$$

(8.72)

- For each 4-valent vertex, write a factor of $-i\lambda$.
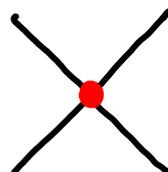
$$\to -i\lambda$$

(8.73)

8.15

- Multiply by the appropriate symmetry factor, i.e. divide by the number of discrete symmetries of the graph.

General Models. We observe that the Feynman graphs and rules for a QFT model reflect quite directly the content of its action:

- In particular, the free part of the action S_0 determines the types and features of the fields and particles. These are reflected by the Feynman propagator G_F which is associated to the edges.
- The interaction part of the action S_{int} contains all the information about the set of interaction vertices.

Examples. Let us apply the Feynman rules to compute the mathematical expressions for a few Feynman graphs.

Consider first the graph for the leading connected contribution $F_{conn}^{(1)}$ to the four-point function.

$$\tag{8.74}$$

Applying the rules for position space, we obtain right away

$$F_{conn}^{(1)} = -i\lambda \int d^4y \prod_{j=1}^{4} G_F(x_j - y). \tag{8.75}$$

In momentum space, the corresponding result is

$$F_{conn}^{(1)} = -i\lambda(2\pi)^4 \delta^4(p_1 + p_2 + p_3 + p_4) \prod_{j=1}^{4} \frac{1}{p_j^2 + m^2 - i\epsilon}. \tag{8.76}$$

This expression is merely a rational function and does not contain any integrals. It is therefore conceivably simpler than its position space analog. Unfortunately, it is generally not easy to perform the Fourier transformation to position space.[6]

Next, consider a slightly more complicated example involving an internal loop.

$$\tag{8.77}$$

---

[6] A notable exception is the massless case where the correlation functions in position space has a reasonably simple form.

Evaluation of the Feynman graph in position space is straight-forward

$$F = \frac{1}{2}(-i\lambda)^2(-i)^6 \int d^4y_1 \, d^4y_2 \, G_F(y_1 - y_2)^2$$
$$\cdot G_F(x_1 - y_1)G_F(x_2 - y_1)G_F(x_3 - y_2)G_F(x_4 - y_2). \qquad (8.78)$$

The symmetry factor is $1/2$ because the two lines of the internal loop can be interchanged.

For momentum space, we first have to label the remaining lines along the internal loop: The total flow of momentum *into* the left vertex from the external lines is $p_1 + p_2$, whereas the momenta on both internal lines are yet undetermined. The sum of internal momenta flowing into the vertex must therefore equal $-p_1 - p_2$ by momentum conservation.[7] One internal momentum remains undetermined, let us call it ℓ and eventually integrate over it. The other one must equal $\ell' = p_1 + p_2 - \ell$.

$$F = \frac{1}{2}(-i\lambda)^2(-i)^6(2\pi)^4\delta^4(p_1 + p_2 + p_3 + p_4)\prod_{j=1}^{4}\frac{1}{p_j + m^2 - i\epsilon}$$
$$\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{\ell^2 + m^2 - i\epsilon}\frac{1}{(p_1 + p_2 - \ell)^2 + m^2 - i\epsilon} . \qquad (8.79)$$

Now we are left with a multiple integral over a rational function. There exist techniques to deal with this sort of problem, we will briefly discuss some of the basic ones at the end of this course. Some integrals like this one can be performed, but most of them remain difficult and it is an art to evaluate them. Unfortunately, numerical methods generally are not applicable either. This is a generic difficulty of QFT with no hope for a universal solution. The Feynman rules are a somewhat formal method and it is hard to extract concrete numbers or functions from them.

8.5 Feynman Rules for QED

Finally, we would like to list the Feynman rules for the simplest physically relevant QFT model, namely quantum electrodynamics (QED). We shall use the Lagrangian in Feynman gauge

$$\mathcal{L}_0 = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \tfrac{1}{2}\partial^\mu A^\nu\partial_\mu A_\nu, \quad \mathcal{L}_{\text{int}} = q\,\bar{\psi}\gamma^\mu\psi\,A_\mu. \qquad (8.80)$$

A non-trivial interacting correlation function in this model must contain as many fermionic fields ψ as conjugates $\bar{\psi}$ due to global U(1) symmetry. Consider such a correlation function

$$\langle A_{\mu_1}(k_1)\ldots A_{\mu_m}(k_m)\,\bar{\psi}_{a_1}(p_1)\psi^{b_1}(q_1)\ldots\bar{\psi}_{a_n}(p_n)\psi^{b_n}(q_n)\rangle. \qquad (8.81)$$

Admissible Feynman graphs have the following properties in addition or instead of to the ones of the ϕ^4 model:

---

[7]By considering the right vertex, it must also equal $p_3 + p_4$. This requirement is consistent by means of overall momentum conservation $p_1 + p_2 + p_3 + p_4 = 0$.

- There are two types of edges: undirected wavy lines (photons) or directed straight lines (electrons and/or positrons).

$$(8.82)$$

- The edges are labelled by a directed flow of 4-momentum ℓ_j.
- The ends of wavy lines are labelled by indices ρ_j and σ_j; the ends of straight lines are labelled by indices c_j and d_j in the direction of the arrow of the straight line.
- There is a 1-valent (external) vertex for each field in the correlator. The momentum inflow and the label at the end of the edge are determined by the corresponding field.

$$(8.83)$$

- There is one type of (internal) vertex: It is 3-valent and connects an ingoing and an outgoing straight line (fermion) with a wavy (photon) line.

$$(8.84)$$

The QED-specific Feynman rules read as follows:

- The graph can have only fermion loops, which contribute an extra factor of (-1) due to their statistics.

$$\to (-1)\int \frac{d^4\ell_j}{(2\pi)^4} \qquad (8.85)$$

- For each wavy edge write a factor of $-i\eta_{\nu_i\nu_j}/(\ell_j^2 - i\epsilon)$.

$$\to \frac{-i\eta_{\nu_i\nu_j}}{\ell_j^2 - i\epsilon} \qquad (8.86)$$

for each straight edge write a factor of $-i(\ell_j\cdot\gamma + m)^{c_j}{}_{d_j}/(\ell_j^2 + m^2 - i\epsilon)$.

$$\to \frac{-i(\ell_j\cdot\gamma + m)^{c_i}{}_{d_j}}{\ell_j^2 + m^2 - i\epsilon} \qquad (8.87)$$

8.18

- For each 3-valent vertex, write a factor of $-iq(\gamma^{\nu_j})^{c_i}{}_{d_i}$.

$$\rightarrow -iq(\gamma^{\nu_j})^{c_i}{}_{d_i} \qquad (8.88)$$

QED and Gauge Invariance.   Note that QED is a gauge theory which requires some gauge fixing. Feynman gauge is very convenient, but any other consistent gauge is acceptable, too. Different gauges imply different propagators which lead to non-unique results for correlation functions. Unique results are only to be expected when the field data within the correlation function is gauge invariant.

More precisely, the correlator should contain the gauge potential $A_\mu(x)$ only in the combination $F_{\mu\nu}(x)$ or as the coupling $\int d^4x\, J^\mu A_\mu$ to some conserved current $J_\mu(x)$. Moreover, charged spinor fields should be combined into uncharged products, e.g. $\bar\psi(x)\ldots\psi(x)$ potentially dressed with covariant derivatives.

9 Particle Scattering

A goal of this course is to understand how to compute scattering processes in particle physics.

9.1 Scattering Basics

Setup. The usual setup for scattering experiments at particle colliders is the following:


$$(9.1)$$
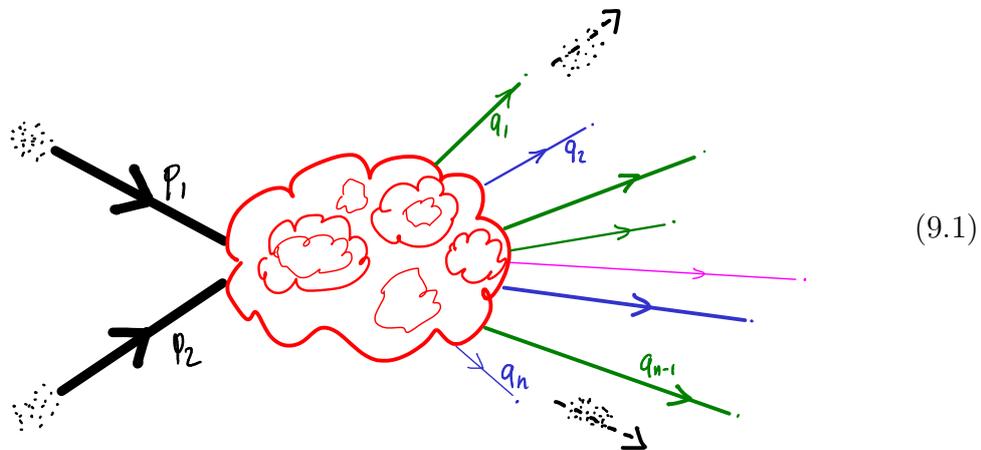
- Two bunches of particles are accelerated to high or relativistic velocities and made to collide.
- Whenever two particles from the bunches come very close, they produce some complicated interacting quantum state.
- After a while this state evolves into several particles moving away in various directions.
- The outgoing particles of each scattering event are measured and recorded.

Some additional comments:

- Quantum mechanics is probabilistic, so a large number of particle collisions must be measured.
- To measure collisions of three or more particles would be technically challenging because they would all have to be focused within a tiny region of space simultaneously. The likelihood for two particles to scatter is much higher.
- By Lorentz symmetry, the directions of the two ingoing momenta $\vec{p}_{1,2}$ can be adjusted arbitrarily. In some reference frame, the momenta will be parallel and along the z-axis. The relevant quantity is the centre of mass energy squared $s = -(p_1 + p_2)^2$. The highest energies $\sqrt{s}$ are obtained where the collisions are head-on with equal but opposite momenta. For practical purposes, the particles can have momenta of different magnitude or one of the two bunches could be a fixed target at rest.

- The particle momenta in the beam are not perfectly aligned. By the uncertainty principle this is actually impossible if the beam is also focused on a finite area.
- The particle detectors are not perfect: They have a certain spatial and temporal resolution. They measure the energy and momenta at a certain resolution. They may not be able to detect and distinguish all kinds of particles; they may miss some particles; they may misidentify some. Scattered particles along the beam direction are hardest to detect.

Cross Sections. How to quantify scattering? Consider a simple classical scattering experiment:

$$(9.2)$$
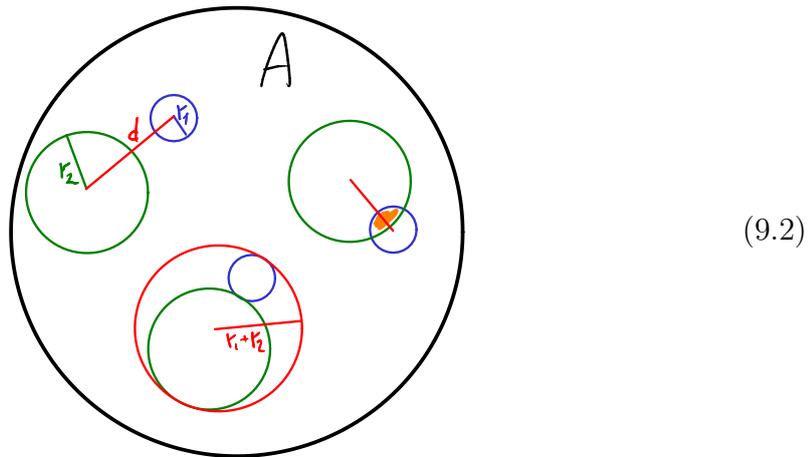
- Take two hard balls of radii r_1, r_2.
- Throw them towards each other along the z axis in opposing directions.
- Depending on the transverse offset d, the balls will either hit $(d < r_1 + r_2)$ or miss $(d > r_1 + r_2)$.[1]
- When the balls hit, they bounce off in different directions.

Quantum mechanics is probabilistic, there cannot be such deterministic output. One has to repeat the experiment many times or perform an experiment with many identical particles and count:

- Accelerate two bunches of n_1 and n_2 particles.
- Focus each bunch on a cross-sectional area of A.
- Repeat the experiment n_{ex} times.
- Count the number of individual scattering events N.

The expectation value for N is

$$N = \frac{n_{\text{ex}} n_1 n_2 \sigma}{A} \, , \qquad (9.3)$$

where the characteristic quantity is the scattering cross section σ. For two classical hard balls one obtains $\sigma = \pi(r_1 + r_2)^2$: Given the transverse position of the first ball, the second ball must be within an area of σ to make the two collide.

In collider experiments one measures scattering cross sections

---

[1]More accurately, at a near miss, the flow of air will also deform the balls' trajectories slightly.

- Total or inclusive cross sections σ simply count the number of overall collision events.
- Differential cross sections $d\sigma$ measure the number of events where the outgoing particles have predetermined momenta.[2] The definition of $d\sigma$ depends on the number of outgoing particles. The so-called phase space is furthermore constrained by Poincaré symmetry.
- One may even resolve the polarisation of the outgoing particles and measure polarised cross sections.

9.2 Cross Sections and Matrix Elements

The computation of the scattering cross section is not straight-forward. Naively, we prepare initial and final states with definite momenta p_1, p_2 and $q_1, \ldots, q_n$ at some times t_{in} and t_{out} in the distant past and distant future

$$\langle \text{f}| \sim \langle q_1, \ldots, q_n|, \qquad |\text{i}\rangle \sim |p_1, p_2\rangle. \tag{9.4}$$

The probability is given by the square of the correlator $\langle \text{f}| \exp(-iH(t_{\text{out}} - t_{\text{in}}))|\text{i}\rangle$

$$\sigma \sim |\langle \text{f}| \exp(-iH(t_{\text{out}} - t_{\text{in}}))|\text{i}\rangle|^2 \tag{9.5}$$

For initial and final states with definite momenta, the correlator contains a delta function $\delta^4(P_{\text{in}} - P_{\text{out}})$ to conserve momentum. It cannot be squared because this would result in a factor of $\delta^4(0) = \infty$. We know that such factors represent some volume of spacetime relevant to the problem. A proper treatment requires the use of wave packets. They actually account for the finite extent of the ingoing bunches, namely the cross-sectional area A, and for the finite resolution of the detector. The factor $\delta^4(0)$ represents this area A among others.

A somewhat tedious calculation in terms of wave packets yields a meaningful result for the differential cross section of $2 \to n$ scattering. At the end of the day, the wave packets can be focused to definite momenta[3]

$$d\sigma = \frac{(2\pi)^4 \delta^4(P_{\text{in}} - P_{\text{out}})}{4|e(\vec{p}_1)\vec{p}_2 - e(\vec{p}_2)\vec{p}_1|} \prod_{k=1}^{n} \frac{d^3\vec{q}_k}{(2\pi)^3 \, 2e(\vec{q}_k)} |M|^2. \tag{9.6}$$

Here M is the appropriate element of the scattering matrix with the momentum-conserving delta function stripped off

$$\lim_{t_{\text{in,out}} \to \mp\infty} \langle \text{f}| \exp(-iH(t_{\text{out}} - t_{\text{in}}))|\text{i}\rangle = (2\pi)^4 \delta^4(P_{\text{in}} - P_{\text{out}}) iM. \tag{9.7}$$

The normalisation is such that in the free theory the correlator for $n = 2$ two final state particles equals[4]

$$2e(\vec{p}_1) \, 2e(\vec{p}_2) \, (2\pi)^6 \delta^3(\vec{p}_1 - \vec{q}_1)\delta^3(\vec{p}_2 - \vec{q}_2). \tag{9.8}$$

---

[2]The direction of the scattered classical balls is determined by the impact parameter d, and hence certain regions of the scattering cross section correspond to specific angles. In quantum mechanics this is mostly a matter of probability.

[3]This quantity is not invariant under Lorentz transformations due to the denominator $|e(\vec{p}_1)\vec{p}_2 - e(\vec{p}_2)\vec{p}_1|$. Nevertheless it transforms like an area as it should.

[4]This contribution representing no scattering is actually removed from M for $2 \to 2$ particle scattering.

The formula simplifies for $2 \to 2$ scattering in the centre of mass frame

$$\frac{d\sigma}{d\Omega} = \frac{1}{4|e(\vec{p}_1)\vec{p}_2 - e(\vec{p}_2)\vec{p}_1|} \frac{|\vec{q}_1|}{16\pi^2\sqrt{s}} |M|^2. \tag{9.9}$$

Here $d\Omega$ represents the spherical angle element of the direction of outgoing particle 1, and $s = -P_{\text{in}}^2 = -(p_1 + p_2)^2$ is the centre of mass energy squared. It becomes even simpler in case all four particles are identical

$$\frac{d\sigma}{d\Omega} = \frac{|M|^2}{64\pi^2 s}. \tag{9.10}$$

9.3 Electron Scattering

We can now compute some realistic scattering event in Quantum Electrodynamics. We shall consider scattering of two electrons into two electrons (Møller scattering).[5]

$$\tag{9.11}$$
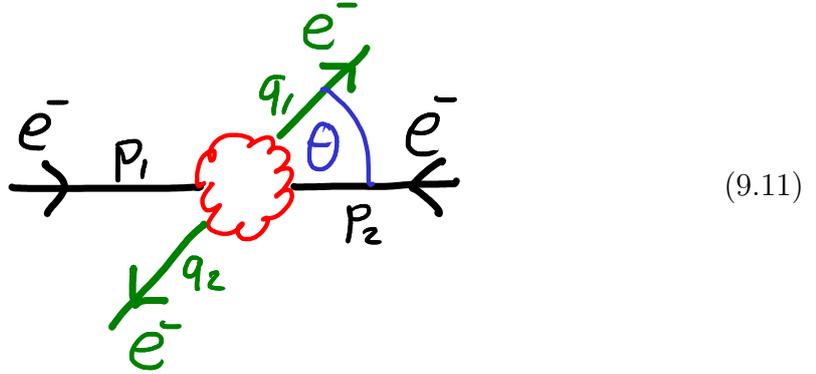
Here, we will not distinguish the two polarisation modes of the electron spin. One might as well consider the polarised cross section, but the experimental setup as well as the theoretical calculation is more challenging.

Initial and Final States.   To prepare the initial and final states we use the interaction picture. The free reference field provides the creation and annihilation operators for the in- and outgoing particles which do not interact when sufficiently far away. Moreover the initial and final states will be practically independent of t_{in} and t_{out} as long as the latter are sufficiently large.

The initial state is composed from two ingoing electrons

$$|\text{i}\rangle = a_\alpha^\dagger(\vec{p}_1) a_\beta^\dagger(\vec{p}_2)|0\rangle. \tag{9.12}$$

The electrons have definite momenta p_1, p_2. Let us assume they are in their centre of mass frame with momenta aligned along the z axis

$$p_{1,2} = (e, 0, 0, \pm p) \tag{9.13}$$

---

[5]Depending on conventions, our calculation may also represent positron-positron scattering. Obviously, the cross section is exactly the same by charge conjugation symmetry.

where $e^2 = p^2 + m^2$. The polarisations α, β are required to set up the state properly. We will not care about them, so we should eventually *average* over all ingoing polarisation configurations.

We want to probe the final state for two outgoing electrons

$$\langle f| = \langle 0|a_\delta(\vec{q}_2)a_\gamma(\vec{q}_1). \tag{9.14}$$

In the centre of mass frame they will escape in two opposite directions with the same magnitude p of momentum. Due to rotational symmetry around the z axis,[6] we only need to probe for particles in the x-z plane

$$q_{1,2} = (e, \pm p \sin\theta, 0, \pm p \cos\theta). \tag{9.15}$$

Again we shall not care about the polarisations. We therefore have to *sum* over all outgoing polarisation configurations.

For a fixed particle momentum p or energy e, we will be interested in the angular distribution of outgoing particles. Due to rotational symmetry the differential cross section $d\sigma/d\Omega$ must be an even function of the scattering angle θ alone. This function also has the symmetry $\theta \to \pi - \theta$ because the outgoing particles are indistinguishable

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega}(\theta) = \frac{d\sigma}{d\Omega}(-\theta) = \frac{d\sigma}{d\Omega}(\pi - \theta). \tag{9.16}$$

Time Evolution. We now insert the time evolution operator U_{int} of the interaction picture between the initial and final states to determine the probability amplitude[7]

$$F = \langle f|U_{\text{int}}(t_{\text{out}}, t_{\text{in}})|i\rangle = (2\pi)^4 \delta^4(P_{\text{in}} - P_{\text{out}})iM. \tag{9.17}$$

The matrix element is a function of the momenta and the polarisations $M_{\alpha\beta\gamma\delta}(p_1, p_2, q_1, q_2)$.[8]

The expansion of the amplitude at leading order reads simply

$$\begin{aligned} F^{(0)} &= \langle f|i\rangle \\ &= 2e(\vec{p}_1)\, 2e(\vec{p}_2)\, (2\pi)^6 \delta^3(\vec{p}_1 - \vec{q}_1)\delta^3(\vec{p}_2 - \vec{q}_2) \\ &\quad - 2e(\vec{p}_1)\, 2e(\vec{p}_2)\, (2\pi)^6 \delta^3(\vec{p}_1 - \vec{q}_2)\delta^3(\vec{p}_2 - \vec{q}_1). \end{aligned} \tag{9.18}$$

The contribution from the free theory represents the situation when the two particles pass by each other without scattering at all. Note that there are two terms corresponding to the fact that the particles are indistinguishable.

At first perturbative order the matrix element vanishes

$$\begin{aligned} F^{(1)} &= i\langle f|S_{\text{int}}|i\rangle = i \int d^4x\, \langle f|\mathcal{L}_{\text{int}}(x)|i\rangle \\ &= iq \int d^4x\, \langle f|A_\mu(x)\bar{\psi}(x)\gamma^\mu\psi(x)|i\rangle = 0 \end{aligned} \tag{9.19}$$

---

[6]We will not measure polarisations which would otherwise break the symmetry.

[7]The conventional factor of i typically makes the leading contributions to M (mostly) real.

[8]We can write it as a function of all the external momenta noting that we shall only evaluate it for $p_1 + p_2 = q_1 + q_2$.

because there is a single electromagnetic field which cannot contract to anything else and thus directly annihilates either of the vacua.

**Second Order.** For the next order we insert two interaction Lagrangians

$$F^{(2)} = \tfrac{1}{2}i^2 \int d^4x\, d^4y\, \langle \text{f}|\text{T}\big(\mathcal{L}_{\text{int}}(x)\mathcal{L}_{\text{int}}(y)\big)|\text{i}\rangle. \tag{9.20}$$

Each of the interaction Lagrangians contains an electromagnetic field. As they would otherwise annihilate the vacua, they have to be contracted via Wick's theorem (for the field A) by a Feynman propagator

$$\tfrac{1}{2}i^2 q^2 \int d^4x\, d^4y\, \langle \text{f}|\text{T}\big(\underline{A_\mu(x)\bar\psi(x)\gamma^\mu\psi(x)\; A_\nu(y)}\bar\psi(y)\gamma^\nu\psi(y)\big)|\text{i}\rangle$$

$$= \tfrac{1}{2}iq^2 \int d^4x\, d^4y\, G^{\text{F}}_{\mu\nu}(x-y)\langle \text{f}|\text{T}\big(\bar\psi(x)\gamma^\mu\psi(x)\; \bar\psi(y)\gamma^\nu\psi(y)\big)|\text{i}\rangle. \tag{9.21}$$

Next, Wick's theorem should be applied to the time-ordered spinor fields yielding several contributions:

- There are two vacuum bubble contributions.

$$\tag{9.22}$$
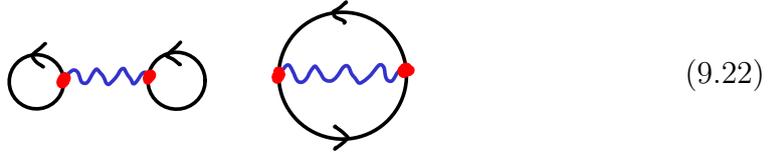
  These vacuum processes take place everywhere and all the time, and they do not interact with the scattering process. As discussed earlier, they must be discarded.
- There are two correction terms with two remaining external fields.

$$\tag{9.23}$$
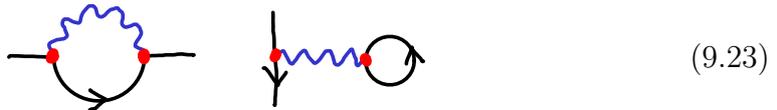
  They contribute to two point functions of spinor fields, but cannot be non-trivial functions of all the four scattering particle momenta. Here they contribute only to forward scattering, and we can safely ignore their contribution. We will discuss their relevance later.
- Finally, there is one connected diagram.

$$\tag{9.24}$$

  This is the leading non-trivial contribution to the scattering process.

Connected Contribution. In our case, the connected diagram is obtained by replacing time ordering by normal ordering

$$\tfrac{1}{2}iq^2 \int d^4x \, d^4y \, G^{\mathrm{F}}_{\mu\nu}(x-y)\langle \mathrm{f}|\mathrm{N}\big(\bar\psi(x)\gamma^\mu\psi(x)\,\bar\psi(y)\gamma^\nu\psi(y)\big)|\mathrm{i}\rangle. \qquad (9.25)$$

Now we need to contract the fields with the external particles. This is achieved by the following two anti-commutators which follow from the mode expansion of the free Dirac field

$$\{\bar\psi(x), a^\dagger_\alpha(\vec p)\} = e^{-ip\cdot x}\bar v_\alpha(\vec p),$$
$$\{a_\alpha(\vec q), \psi(x)\} = e^{iq\cdot x}v_\alpha(\vec q). \qquad (9.26)$$

Putting everything together we obtain the matrix element

$$F^{(2)}_{\mathrm{conn}} = iq^2 \int d^4x \, d^4y \, G^{\mathrm{F}}_{\mu\nu}(x-y)e^{iq_1\cdot x + iq_2\cdot y - ip_1\cdot x - ip_2\cdot y}$$
$$\cdot \bar v_\alpha(\vec p_1)\gamma^\mu v_\gamma(\vec q_1)\,\bar v_\beta(\vec p_2)\gamma^\nu v_\delta(\vec q_2) - \dots$$

$$(9.27)$$

The omitted term takes the same form, but with the two outgoing particles exchanged ($q_1 \leftrightarrow q_2$, $\gamma \leftrightarrow \delta$). The reason for the doubling of terms is that two identical types of particles are scattered. The two-particle wave function is anti-symmetric because the particles are fermionic.

The two remaining integrals are Fourier transforms. One of them transforms the Feynman propagator to momentum space. The other one generates the momentum conserving delta function. Altogether the integrals yields

$$(2\pi)^4\delta^4(P_{\mathrm{out}} - P_{\mathrm{in}})\, G^{\mathrm{F}}_{\mu\nu}(q_1 - p_1). \qquad (9.28)$$

We now separate off the momentum conserving delta function and write the matrix element M [9]

$$M_{\alpha\beta\gamma\delta} = \frac{q^2\eta_{\mu\nu}}{(p_1-q_1)^2}\,\bar v_\alpha(\vec p_1)\gamma^\mu v_\gamma(\vec q_1)\,\bar v_\beta(\vec p_2)\gamma^\nu v_\delta(\vec q_2) - \dots \ . \qquad (9.29)$$

We could try to evaluate the various spinor products. It turns out to be much simpler to square the matrix element first

$$|M|^2 = \frac{1}{4}\sum_{\alpha,\beta,\gamma,\delta} M_{\alpha\beta\gamma\delta}M^*_{\alpha\beta\gamma\delta}. \qquad (9.30)$$

The factor of $1/2^2$ originates from averaging over the polarisations of the ingoing particles.

---

[9]The $i\epsilon$ prescription for the Feynman propagator will not be relevant here.

Sum over Polarisations. A pleasant feature of the polarisation sums is that they can be performed by the completeness relations for spinor solutions

$$\sum_\alpha v_\alpha(\vec{p})\bar{v}_\alpha(\vec{p}) = p\cdot\gamma - m. \tag{9.31}$$

We obtain the following three terms

$$|M|^2 = \frac{q^4 T_{tt}}{4(p_1 - q_1)^4} + \frac{q^4 T_{uu}}{4(p_1 - q_2)^4} - \frac{q^4 T_{tu}}{2(p_1 - q_1)^2(p_1 - q_2)^2}, \tag{9.32}$$

corresponding to the diagrams

$$\tag{9.33}$$
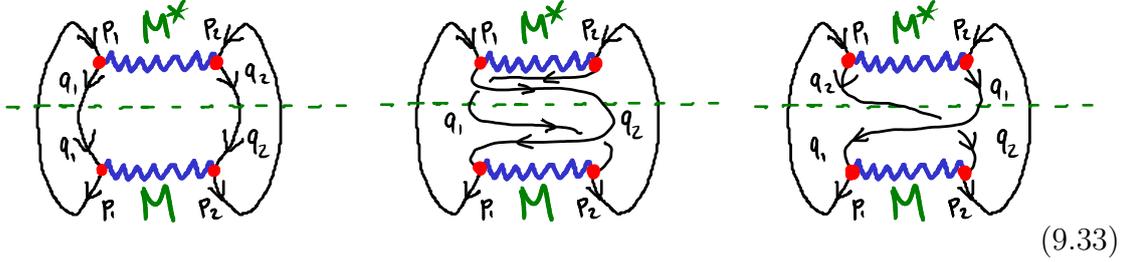
The spinor products have turned into the traces

$$
\begin{aligned}
T_{tt} &= \operatorname{tr}[(p_1\cdot\gamma - m)\gamma_\mu(q_1\cdot\gamma - m)\gamma_\nu] \\
&\quad \cdot \operatorname{tr}[(p_2\cdot\gamma - m)\gamma^\mu(q_2\cdot\gamma - m)\gamma^\nu], \\
T_{uu} &= \operatorname{tr}[(p_1\cdot\gamma - m)\gamma_\mu(q_2\cdot\gamma - m)\gamma_\nu] \\
&\quad \cdot \operatorname{tr}[(p_2\cdot\gamma - m)\gamma^\mu(q_1\cdot\gamma - m)\gamma^\nu], \\
T_{tu} &= \operatorname{tr}[(p_1\cdot\gamma - m)\gamma_\mu(q_1\cdot\gamma - m)\gamma^\nu(p_2\cdot\gamma - m)\gamma^\mu(q_2\cdot\gamma - m)\gamma_\nu].
\end{aligned} \tag{9.34}
$$

The double-trace terms are most conveniently evaluated using the spinor trace formulas[10]

$$
\begin{aligned}
\operatorname{tr}(1) &= 4, \\
\operatorname{tr}(\gamma^\mu) &= 0, \\
\operatorname{tr}(\gamma^\mu\gamma^\nu) &= -4\eta^{\mu\nu}, \\
\operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^\rho) &= 0, \\
\operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}).
\end{aligned} \tag{9.35}
$$

This brings T_{tt} into the following form

$$
\begin{aligned}
T_{tt} &= 16(p_{1\mu}q_{1\nu} + q_{1\mu}p_{1\nu} - (p_1\cdot q_1 + m^2)\eta_{\mu\nu}) \\
&\quad \cdot (p_2^\mu q_2^\nu + q_2^\mu p_2^\nu - (p_2\cdot q_2 + m^2)\eta^{\mu\nu}) \\
&= 32(p_1\cdot p_2)(q_1\cdot q_2) + 32(p_1\cdot q_2)(q_1\cdot p_2) \\
&\quad + 32m^2(p_1\cdot q_1 + p_2\cdot q_2) + 64m^4.
\end{aligned} \tag{9.36}
$$

---

[10]The latter of these formulas follow from anti-commuting one gamma matrix past all the others.

The other double-trace term takes a similar form with q_1 and q_2 interchanged. The crossed single-trace term can be simplified by means of the enveloping identities

$$\gamma_\mu \gamma^\mu = -4,$$
$$\gamma_\mu \gamma^\nu \gamma^\mu = 2\gamma^\nu,$$
$$\gamma_\mu \gamma^\nu \gamma^\rho \gamma^\mu = 4\eta^{\nu\rho},$$
$$\gamma_\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\mu = 2\gamma^\sigma \gamma^\rho \gamma^\nu. \tag{9.37}$$

After some algebra we obtain

$$\begin{aligned} T_{tu} = & -32(q_1 \cdot q_2)(p_1 \cdot p_2) - 32m^4 \\ & -16m^2(p_1 \cdot p_2 + p_1 \cdot q_1 + p_1 \cdot q_2 + p_2 \cdot q_1 + p_2 \cdot q_2 + q_1 \cdot q_2). \end{aligned} \tag{9.38}$$

Mandelstam Invariants.   In order to simplify the expressions we introduce the Mandelstam invariants

$$s = -(p_1 + p_2)^2 = -(q_1 + q_2)^2,$$
$$t = -(p_1 - q_1)^2 = -(p_2 - q_2)^2,$$
$$u = -(p_1 - q_2)^2 = -(p_2 - q_1)^2. \tag{9.39}$$

Inverting the relations we can write all scalar products of momenta using the s, t, u

$$p_1 \cdot p_2 = q_1 \cdot q_2 = m^2 - \tfrac{1}{2}s,$$
$$p_1 \cdot q_1 = p_2 \cdot q_2 = \tfrac{1}{2}t - m^2,$$
$$p_1 \cdot q_2 = p_2 \cdot q_1 = \tfrac{1}{2}u - m^2. \tag{9.40}$$

Note furthermore that momentum conservation implies the relation[11]

$$s + t + u = 4m^2. \tag{9.41}$$

Using Mandelstam invariants, the traces can be expressed very compactly as[12]

$$T_{tt} = 8(t^2 - 2su + 8m^4),$$
$$T_{uu} = 8(u^2 - 2st + 8m^4),$$
$$T_{tu} = -8(s^2 - 8m^2 s + 12m^4). \tag{9.42}$$

The squared matrix element now reads[13]

$$|M|^2 = q^4 \left(\frac{u-s}{t} + \frac{t-s}{u} \right)^2 + \frac{16q^4 m^2 (5m^2 - 2s)}{tu}. \tag{9.43}$$

This expression is symmetric under exchange of t and u as it should because the outgoing particles are of the same kind.

---

[11]This constraint implies that functions of s, t, u can be written in several alternative ways much alike functions of p_1, p_2, q_1, q_2 which are constrained by $p_1 + p_2 - q_1 - q_2 = 0$.

[12]It is not straight-forward to derive these particular expressions, but it is easy to confirm that they match with some other expression upon substituting, e.g. $s = 4m^2 - t - u$.

[13]We can identify the first term as the corresponding result in scalar QED.

Angular Distribution. In order to understand the angular distribution of scattered particles we express the invariants in terms of the scattering angle θ

$$s = 4p^2 + 4m^2, \quad t = -2p^2(1 - \cos\theta), \quad u = -2p^2(1 + \cos\theta). \tag{9.44}$$

and insert everything into the differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{q^4}{64\pi^2 e^2} \left(\frac{(4p^2 + 2m^2)^2}{p^4 \sin^4\theta} - \frac{8p^4 + 12m^2p^2 + 3m^4}{p^4 \sin^2\theta} + 1 \right). \tag{9.45}$$

This expression is the leading non-trivial contribution to the angular distribution of scattered electrons.

We can however notice a problem by inspecting the expression. It diverges when

- the electron momenta p are small or
- the scattering angle θ is close to 0 or π.

In those regimes the formula cannot be trusted. The deeper reason for the divergences is that the photons which transmit the electromagnetic force are massless. Massless particles cause some conceptual problems in scattering processes.

The divergences are also relevant to the total cross section[14]

$$\sigma = \int_0^1 2\pi d\cos\theta \, \frac{d\sigma}{d\Omega}. \tag{9.46}$$

This integral diverges at $\cos\theta = 1$.

In order to properly address the above problematic regimes, one would have to take higher perturbative corrections and competing processes into account. However, only the full non-perturbative expression can provide exact results in those regimes.

Nevertheless one should not expect a meaningful result for the total cross section because the electromagnetic force is long-ranged: The photon propagator is not exponentially suppressed at long distances. Effectively all particles scatter at least by tiny amount and therefore the overall probability for scattering is 1. The scattering cross section σ is the complete area A of the bunches which is infinite due to our assumption of exactly defined momenta.

---

[14]The outgoing particles are indistinguishable, hence the integration extends only over one half of the spherical angles. This is sufficient since at leading order $\langle f|i\rangle$ has two terms one of which covers the opposite angles $\pi - \theta$. Alternative the integral over all spherical angles must be multiplied by a factor of $\frac{1}{2}$.

Crossing Symmetry. A closely related process is the scattering of electrons and positrons (Bhabha scattering).

$$(9.47)$$
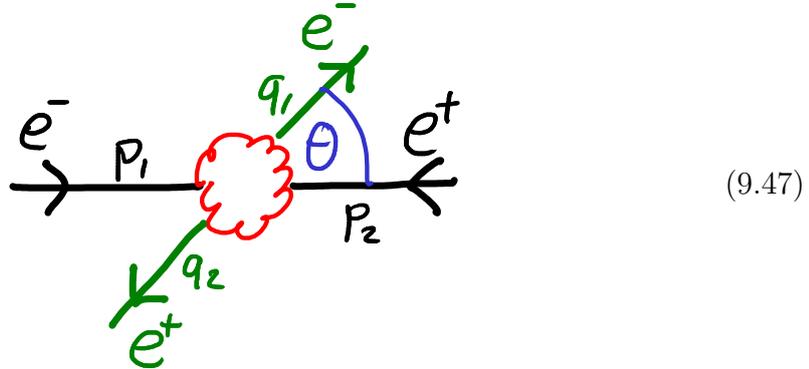
It can be computed in much the same way.

The relevant connected diagrams for electron-positron scattering are

$$(9.48)$$
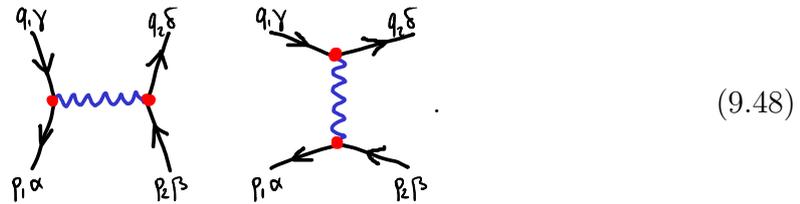
The cross section turns out to be exactly the same as for electron scattering but with s and u interchanged

$$s \leftrightarrow u. \qquad (9.49)$$

The resulting leading contribution to the squared matrix element is[15]

$$|M|^2 = q^4 \left(\frac{s-u}{t} + \frac{t-u}{s} \right)^2 + \frac{16q^4 m^2(5m^2 - 2u)}{st} . \qquad (9.50)$$

Indeed the computation is exactly the same when replacing

$$p_2 \leftrightarrow -q_2, \qquad \sum_\alpha v_\alpha(\vec{p_2})\bar{v}_\alpha(\vec{p_2}) \leftrightarrow -\sum_\alpha u_\alpha(\vec{q_2})\bar{u}_\alpha(\vec{q_2}). \qquad (9.51)$$

This relationship is called crossing symmetry. In terms of Feynman diagrams, the positron line is equivalent to the electron line in reverse direction


$$(9.52)$$

---
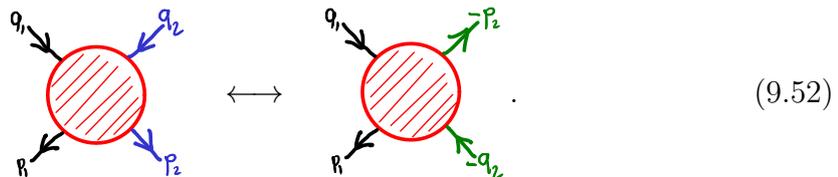
[15]Apart from effects due to identical particles, the electron-positron scattering cross section does not differ substantially from the case of electron-electron scattering. The difference between attraction and repulsion manifests in the phase of matrix elements rather than in their absolute value.

9.4   Pair Production

The above electron-positron scattering involves a process where the two particles combine into a photon and subsequently split up into a pair. This process is mixed with photon exchange in the t channel.

A class of similar processes is pair production where the oppositely charged particles annihilate and create a pair of charged particles of a different kind.

$$(9.53)$$
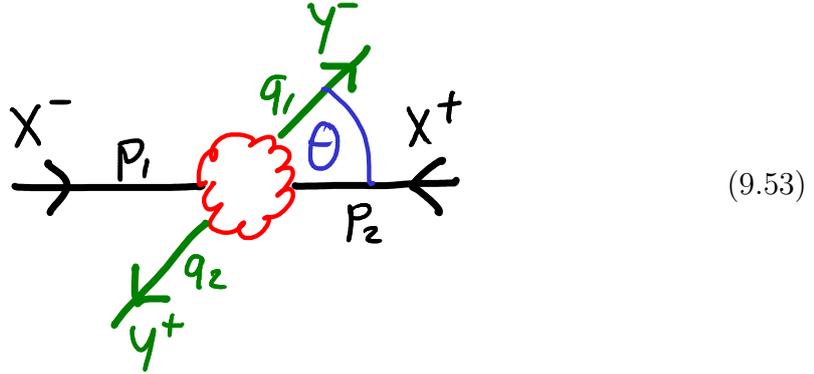
Let us compute scattering cross sections for such processes.

We assume that the outgoing particles have a mass m_f and charge $\pm q_f$ which is different from the ingoing ones labelled by m_i and $\pm q_i$.

Spinor Processes.   First we consider the case where all external particles are spinors

$$(9.54)$$

There is now only one spinor trace to be evaluated

$$
\begin{aligned}
T =\ & \mathrm{tr}[(p_1\!\cdot\!\gamma - m_i)\gamma_\mu(p_2\!\cdot\!\gamma + m_i)\gamma_\nu] \\
& \cdot \mathrm{tr}[(q_1\!\cdot\!\gamma + m_f)\gamma^\mu(q_2\!\cdot\!\gamma - m_f)\gamma^\nu], \\
=\ & 16(-p_{1\mu}p_{2\nu} - p_{2\mu}p_{1\nu} - (-p_1\!\cdot\!p_2 + m_i^2)\eta_{\mu\nu}) \\
& \quad \cdot (-q_1^\mu q_2^\nu - q_2^\mu q_1^\nu - (-q_1\!\cdot\!q_2 + m_f^2)\eta^{\mu\nu}) \\
=\ & 4(t-u)^2 + 4s^2 + 16(m_i^2 + m_f^2)s.
\end{aligned}
\qquad (9.55)
$$

The Mandelstam invariants are defined as above, but due to the different masses their relationships have to be adjusted

$$
\begin{aligned}
p_1\!\cdot\!p_2 &= m_i^2 - \tfrac{1}{2}s, \\
q_1\!\cdot\!q_2 &= m_f^2 - \tfrac{1}{2}s, \\
p_1\!\cdot\!q_1 = p_2\!\cdot\!q_2 &= \tfrac{1}{2}t - \tfrac{1}{2}m_i^2 - \tfrac{1}{2}m_f^2, \\
p_1\!\cdot\!q_2 = p_2\!\cdot\!q_1 &= \tfrac{1}{2}u - \tfrac{1}{2}m_i^2 - \tfrac{1}{2}m_f^2, \\
s + t + u &= 2m_i^2 + 2m_f^2.
\end{aligned}
\qquad (9.56)
$$

Next we express the invariants as functions of the scattering angle

$$s = 4e^2, \quad t = -2p_\mathrm{i}p_\mathrm{f}(1 - \cos\theta), \quad u = -2p_\mathrm{i}p_\mathrm{f}(1 + \cos\theta). \qquad (9.57)$$

The unpolarised matrix element squared now reads

$$
\begin{aligned}
|M|^2 &= \frac{q_\mathrm{i}^2 q_\mathrm{f}^2 T}{4s} \\
&= q_\mathrm{i}^2 q_\mathrm{f}^2 \left(\frac{e^2 - m_\mathrm{i}^2}{e^2} \frac{e^2 - m_\mathrm{f}^2}{e^2} \cos^2\theta + \frac{m_\mathrm{i}^2 + m_\mathrm{f}^2}{e^2} + 1 \right).
\end{aligned}
\qquad (9.58)
$$

The formula for the differential cross section for our configuration of masses and momenta in the centre of mass frame reads

$$\frac{d\sigma}{d\Omega} = \sqrt{\frac{e^2 - m_\mathrm{f}^2}{e^2 - m_\mathrm{i}^2}} \frac{|M|^2}{256\pi^2 e^2}. \qquad (9.59)$$

Total Cross Section. This expression is free from singularities and can be integrated to a total cross section

$$\sigma = 4\pi \int_{-1}^{1} \frac{d\cos\theta}{2} \frac{d\sigma}{d\Omega} = \frac{1}{64\pi e^2} \sqrt{\frac{e^2 - m_\mathrm{f}^2}{e^2 - m_\mathrm{i}^2}} \int_{-1}^{1} \frac{d\cos\theta}{2} |M|^2. \qquad (9.60)$$

Upon integration we obtain the final result

$$\sigma = \frac{q_\mathrm{i}^2 q_\mathrm{f}^2}{48\pi e^2} \sqrt{\frac{e^2 - m_\mathrm{f}^2}{e^2 - m_\mathrm{i}^2}} \frac{e^2 + \frac{1}{2}m_\mathrm{i}^2}{e^2} \frac{e^2 + \frac{1}{2}m_\mathrm{f}^2}{e^2}. \qquad (9.61)$$

We can plot the energy-dependence of this function.

$$(9.62)$$
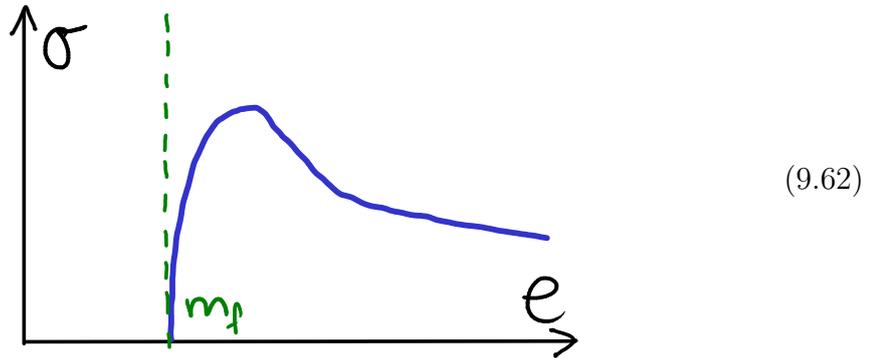
Quite clearly the total energy $2e$ of the scattered particles must be at least as large as the sum of masses $2m_\mathrm{f}$ of produced particles. There is a sharp increase above production threshold, a maximum slightly above threshold (for $m_\mathrm{i} < m_\mathrm{f}$), and a slow $1/e^2$ descent.

Processes Involving Scalars. It is interesting to compare this process to the corresponding one of charged scalars. The matrix element reads

$$|M|^2 = q_{\mathrm{i}}^2 q_{\mathrm{f}}^2 \frac{(t-u)^2}{s^2} = q_{\mathrm{i}}^2 q_{\mathrm{f}}^2 \frac{e^2 - m_{\mathrm{i}}^2}{e^2} \frac{e^2 - m_{\mathrm{f}}^2}{e^2} \cos^2 \theta \,, \tag{9.63}$$

and after integration we obtain the total cross section

$$\sigma = \frac{q_{\mathrm{i}}^2 q_{\mathrm{f}}^2}{192 \pi e^2} \sqrt{\frac{e^2 - m_{\mathrm{f}}^2}{e^2 - m_{\mathrm{i}}^2}} \frac{e^2 - m_{\mathrm{i}}^2}{e^2} \frac{e^2 - m_{\mathrm{f}}^2}{e^2} \,. \tag{9.64}$$

Let us finally consider a mixed process of spinors scattering into scalars for which the matrix element reads

$$\begin{aligned} |M|^2 &= q_{\mathrm{i}}^2 q_{\mathrm{f}}^2 \frac{-(t-u)^2 + s^2 - 4m_{\mathrm{f}}^2 s}{s^2} \\ &= q_{\mathrm{i}}^2 q_{\mathrm{f}}^2 \frac{e^2 - (e^2 - m_{\mathrm{i}}^2)\cos^2 \theta}{e^2} \frac{e^2 - m_{\mathrm{f}}^2}{e^2} \,. \end{aligned} \tag{9.65}$$

For the total cross section we obtain

$$\sigma = \frac{q_{\mathrm{i}}^2 q_{\mathrm{f}}^2}{192 \pi e^2} \sqrt{\frac{e^2 - m_{\mathrm{f}}^2}{e^2 - m_{\mathrm{i}}^2}} \frac{e^2 + \frac{1}{2} m_{\mathrm{i}}^2}{e^2} \frac{e^2 - m_{\mathrm{f}}^2}{e^2} \,. \tag{9.66}$$

The opposite process of scalars scattering into spinors merely has a different overall factor

$$\sigma = \frac{q_{\mathrm{i}}^2 q_{\mathrm{f}}^2}{48 \pi e^2} \sqrt{\frac{e^2 - m_{\mathrm{f}}^2}{e^2 - m_{\mathrm{i}}^2}} \frac{e^2 - m_{\mathrm{i}}^2}{e^2} \frac{e^2 + \frac{1}{2} m_{\mathrm{f}}^2}{e^2} \,. \tag{9.67}$$

We observe that, although the matrix elements differ substantially, the final cross sections take a very predictable form: There are particular factors for scalars and spinors in the initial and final states, namely

- $e^2 - m^2$ for ingoing scalars,
- $e^2 + \frac{1}{2} m_{\mathrm{i}}^2$ for ingoing spinors,
- $e^2 - m_{\mathrm{f}}^2$ for outgoing scalars,
- $4(e^2 + \frac{1}{2} m_{\mathrm{f}}^2)$ for outgoing spinors.

The square root on the other hand is a kinematical factor corresponding to the number/volume of initial and final states (phase space).

The factors actually follow from the total spin of the pairs of particles. Assume that a spin-0 state couples to the photon by a factor $e^2 - m^2$ whereas a spin-1 state couples via $e^2 + m^2$.[16] Then for scalar we immediately obtain $e^2 - m^2$ whereas the four polarisations of two spinors make up one spin-0 and three spin-1 states yielding a factor of $(e^2 - m^2) + 3(e^2 + m^2) = 4(e^2 + \frac{1}{2} m^2)$. For ingoing spinors the factor of 4 is compensated by taking the average rather than that sum.

---

[16]It is reasonable that close to threshold $e^2 = m^2$ the spin-1 coupling dominates because the photon is a vector particle. Above threshold the outgoing particles can also have orbital angular momentum whose spin-1 component would also couple to the photon. Therefore the increase at threshold is much softer for scalars than for spinors.

9.5 Loop Contributions

We have obtained the leading-order contributions to some particle scattering processes. Let us finally peek at contributions at higher orders in the perturbation series.

For the electron scattering process at the next order q^4 there are several types of diagrams contributing.
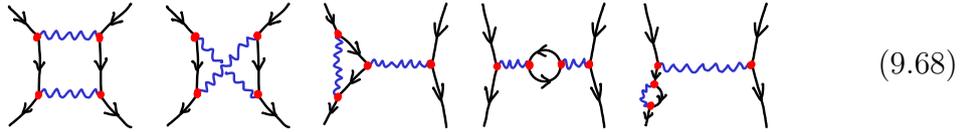

$$(9.68)$$

Here we listed only the connected diagrams up to obvious symmetric copies.

It is easy to see that these diagrams lead to two types of problems

- The diagram with a bubble on the external leg is ill-defined.
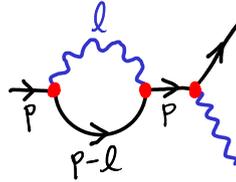
$$(9.69)$$

  Due to momentum conservation the momentum on both sides of the bubble is the same. All external momenta originate from particle creation and annihilation operators $a^\dagger(\vec{p})$ and $a(\vec{p})$. These momenta are exactly on the mass shell $p^2 = -m^2$. Conversely, the internal line represents a Feynman propagator $1/(p^2 + m^2 - i\epsilon)$ which is to be evaluated right on the pole

$$\frac{1}{p^2 + m^2 - i\epsilon} \quad \text{at } p^2 = -m^2. \tag{9.70}$$

  This diagram therefore makes no sense as a contribution to the scattering process.
- Most of the integrals are actually divergent in the UV, i.e. where the loop momentum ℓ is very large. For example, the bubble on the photon line yields

 $\sim \int d^4\ell \, \frac{\ell^2 + \dots}{\ell^4 + \dots} \sim \int \frac{d^4\ell}{\ell^2} \to \infty.$

$$(9.71)$$

We have to understand how to deal with these two problems. This is going to be the subject of the final two chapters.

10   Scattering Matrix

When we computed some simple scattering processes we did not really know what we were doing. At leading order this did not matter, but at higher orders complications arise. Let us therefore discuss the asymptotic particle states and their scattering matrix in more detail.

10.1   Asymptotic States

First we need to understand asymptotic particle states in the interacting theory

$$|p_1, p_2, \ldots\rangle. \tag{10.1}$$

In particular, we need to understand how to include them in calculations by expressing them in terms of the interacting field $\phi(x)$.

Asymptotic particles behave like free particles at least in the absence of other nearby asymptotic particles. For free fields we have seen how to encode the particle modes into two-point correlators, commutators and propagators. Let us therefore investigate these characteristic functions in the interacting model.

Two-Point Correlator.   Consider first the correlator of two interacting fields

$$\Delta_+(x - y) := \langle 0|\phi(x)\phi(y)|0\rangle. \tag{10.2}$$

Due to Poincaré symmetry, it must take the form

$$\Delta_+(x) = \int \frac{d^4p}{(2\pi)^4} \, e^{-ip\cdot x} \, \theta(p_0)\rho(-p^2). \tag{10.3}$$

The factor $\theta(p_0)$ ensures that all excitations of the ground state $|0\rangle$ have positive energy. The function $\rho(s)$ parametrises our ignorance. We do not want tachyonic excitations, hence the function should be supported on positive values of $s = m^2$. We now insert a delta function to express the correlator in terms of the free two-point correlator $\Delta_+(s; x, y)$ with mass $\sqrt{s}$ (Källén, Lehmann)

$$\begin{aligned}
\Delta_+(x) &= \int_0^\infty \frac{ds}{2\pi} \, \rho(s) \int \frac{d^4p}{(2\pi)^4} \, e^{-ip\cdot x} \, \theta(p_0)2\pi\delta(p^2 + s) \\
&= \int_0^\infty \frac{ds}{2\pi} \, \rho(s)\Delta_+(s; x).
\end{aligned} \tag{10.4}$$

This identifies $\rho(s)$ as the spectral function for the field $\phi(x)$: It tells us by what amount particle modes of mass $\sqrt{s}$ will be excited by the field $\phi(x)$.[1]

---

[1]The spectral function describes the spectrum of quantum states only to some extent. However, not all states may be excited by the action of a single $\phi(x)$. In particular, in a model with several fields, each field can excite only a subset of particles or states (e.g. the appropriate charges have to match).

Spectral Function. For a free field of mass m_0 we clearly have

$$\rho_0(s) = 2\pi\delta(s - m_0^2). \qquad (10.5)$$

For weakly interacting fields, we should obtain a similar expression. In typical situations we expect the spectral function to have the following shape

$$(10.6)$$
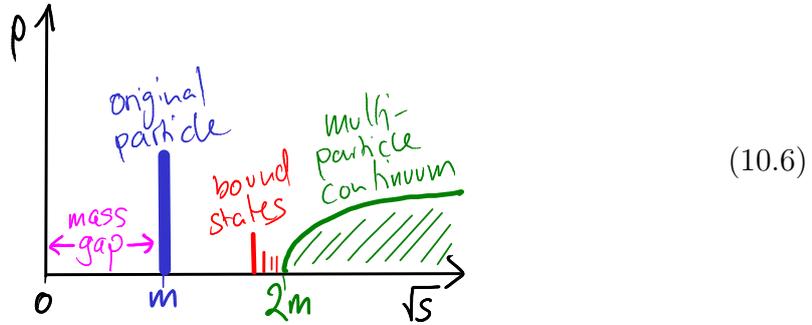
The sharp isolated peak represents a single particle excitation with mass m. Now the field $\phi(x)$ may also excite multi-particle modes (with the same quantum numbers). Multi-particle modes in the free theory would have energy $e \geq 2m$. In the spectral function they form a continuum since the momenta of the individual particles can sum up to arbitrary energies in the frame at rest. In the presence of interactions, bound states may form whose rest energies are somewhat below $e = 2m$. Whenever these bound states are stable they will also be represented by sharp peaks.

We observe that our spectral function has at least two mass gaps: One separates the vacuum from the lowest excitation; the other separates this latter from bound states and the multi-particle continuum. The isolated modes are called asymptotic particles. This is the type of particle which we would like to collide. The assumption of a mass gap is crucial in this definition.

For weak interactions, we expect that the free particle mode approximates the asymptotic particle well.[2] The interactions may shift the mass $m_0 \to m$ slightly; they may also change the strength with which this mode is excited by the field $\phi(x)$. Therefore the weakly interacting spectral function takes the form

$$\rho(s) = 2\pi Z\delta(s - m^2) + \text{bound states} + \text{continuum}. \qquad (10.7)$$

The factor Z is called field strength or wave function renormalisation.

Asymptotic Particles. Based on the above discussion we can expand the field $\phi(x)$ as

$$\phi(x) = \underbrace{\sqrt{Z}\phi_{\text{as}}(x)}_{a^\dagger + a} + \underbrace{\text{bound states} + \text{continuum}}_{(a^\dagger)^n + a^n} + \underbrace{\text{operators}}_{(a^\dagger)^m a^n}. \qquad (10.8)$$

---

[2]For reasonably strong interactions, bound states may approach the single particle states and even acquire lower energies. This case shows that the notion of fundamental particles is not evident in general QFT, but it belongs to weakly interacting models. In fact, some models may have alternative formulations where the fundamental degrees of freedom are some bound states of the original formulation.

Here $\phi_{\mathrm{as}}(x)$ is a canonically normalised free field of mass m expressed by means of creation and annihilation operators $a^\dagger, a$

$$\phi_{\mathrm{as}}(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 \, 2e(\vec{p})} \big(e^{-ip\cdot x} a(\vec{p}) + e^{ip\cdot x} a^\dagger(\vec{p}) \big). \tag{10.9}$$

The other terms in the field $\phi(x)$ are multiple creation and/or annihilation operators.

Single particle asymptotic states are created simply by $a^\dagger(\vec{p})$ from the vacuum. The Hamiltonian H_{as} for the free asymptotic field reads

$$H_{\mathrm{as}} = \int \frac{d^3\vec{p}}{(2\pi)^3 \, 2e(\vec{p})} \, e(\vec{p}) a^\dagger(\vec{p}) a(\vec{p}). \tag{10.10}$$

The characteristic property of H_{as} is that it reproduces exactly the time evolution of the vacuum and single-particle states

$$H_{\mathrm{as}}|0\rangle = 0 = H|0\rangle, \quad H_{\mathrm{as}} a^\dagger(\vec{p})|0\rangle = e(\vec{p}) a^\dagger(\vec{p})|0\rangle = H a^\dagger(\vec{p})|0\rangle. \tag{10.11}$$

We shall use the free creation and annihilation operators as some convenient basis to expand our interacting fields. The omitted terms in the field $\phi(x)$ are some higher-order polynomials in the operators $a^\dagger, a$ which create and annihilate bound state particles and states from the multi-particle continuum.

Commutator and Normalisation. The other characteristic functions now follow from our expression for the correlator. As before these can be expressed as convolutions of the same spectral function $\rho(s)$ with their free counterparts.

The expectation value of the unequal time commutator

$$\Delta(x - y) := \langle 0|[\phi(\vec{x}), \phi(\vec{y})]|0\rangle \tag{10.12}$$

therefore reads

$$\Delta(x) = \int_0^\infty \frac{ds}{2\pi} \, \rho(s) \Delta(s; x). \tag{10.13}$$

We know that for a normalised free field the equal time commutation relations imply $-\dot{\Delta}(s; 0, \vec{x}) = i\delta^3(\vec{x})$. Hence

$$\langle 0|[\phi(\vec{x}), \dot{\phi}(\vec{y})]|0\rangle = -\dot{\Delta}(0, \vec{x} - \vec{y}) = i\delta^3(\vec{x}) \int_0^\infty \frac{ds}{2\pi} \, \rho(s). \tag{10.14}$$

Assuming that the field $\phi(x)$ is canonically normalised,[3] we have the constraint

$$\int_0^\infty \frac{ds}{2\pi} \, \rho(s) = 1. \tag{10.15}$$

When using the above expansion of the real field $\phi(x)$ in terms of creation and annihilation operators, it also follows that the function $\rho(s)$ must be positive. Hence the coefficient Z for the asymptotic modes should be between 0 and 1.

---

[3]This is evident at least if the interaction terms do not contain derivatives.

10.2 S-Matrix

For the scattering setup we define two asymptotic regions of spacetime, one in the distance past $t_{\text{in}} \to -\infty$ and one in the distant future $t_{\text{out}} \to +\infty$.

Asymptotic Regions. On the initial time slice we create wave packets which are well separated in position space and narrowly peaked in momentum space. We let these quantum mechanical wave packets evolve in time. At some instance the wave packets collide. Then the state is evolved further until all outgoing wave packets are sufficiently well separated

$$|\text{f}\rangle = \exp\big(-iH(t_{\text{out}} - t_{\text{in}})\big)|\text{i}\rangle. \tag{10.16}$$

Now the initial and final states are in the Schrödinger picture and they evolve even at asymptotic times. It is hard to compare them to see what the effect of scattering is.[4]

$$\tag{10.17}$$
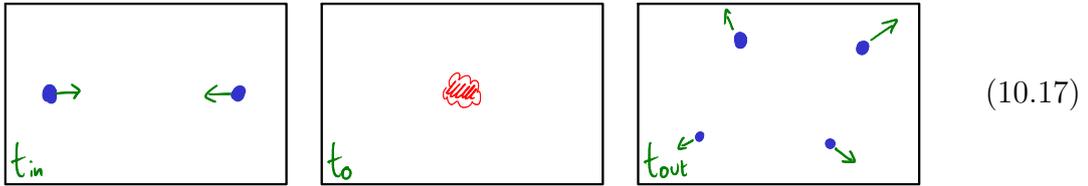
At asymptotic times the wave packets are assumed to be sufficiently well separated such that they effectively do not interact. Therefore we can use the asymptotic Hamiltonian of the asymptotic field ϕ_{as} [5]

$$H_{\text{as}} = \int \frac{d^3\vec{p}}{(2\pi)^3 \, 2e(\vec{p})} \, e(\vec{p}) a^\dagger(\vec{p}) a(\vec{p}). \tag{10.18}$$

to shift the two time slices onto a common one conventionally positioned at $t = 0$

$$|\text{out}\rangle = \exp\big(iH_{\text{as}}t_{\text{out}}\big)|\text{f}\rangle, \qquad |\text{i}\rangle = \exp\big(-iH_{\text{as}}t_{\text{in}}\big)|\text{in}\rangle. \tag{10.19}$$

The relationship between the in and out states is the following

$$\begin{aligned}
|\text{out}\rangle &= \exp\big(iH_{\text{as}}t_{\text{out}}\big) \exp\big(-iH(t_{\text{out}} - t_{\text{in}})\big) \exp\big(-iH_{\text{as}}t_{\text{in}}\big)|\text{in}\rangle \\
&=: U_{\text{as}}(t_{\text{out}}, t_{\text{in}})|\text{in}\rangle.
\end{aligned} \tag{10.20}$$

The in and out states $|\text{in}\rangle$ and $|\text{out}\rangle$ are both defined at time $t = 0$. Consequently, they are elements of the same Hilbert space and can be compared directly. The

---

[4]The latter figure is somewhat misleading in a quantum mechanical setting. It shows only one out of many potential final states.

[5]This asymptotic Hamiltonian is a specialisation of the free Hamiltonian H_0 used previously in the interaction picture. The free Hamiltonian was merely required to agree with the full Hamiltonian at leading order. The asymptotic Hamiltonian furthermore has to agree with the full Hamiltonian exactly when action on the vacuum or one-particle states.

operator U_{as} is the time evolution operator for the interaction picture based on the asymptotic Hamiltonian H_{as} and the reference time slice at $t = 0$.

$$(10.21)$$

S-Matrix Definition.   As interactions have become negligible at asymptotic times, the in and out states are almost independent of t_{in} and t_{out}. It therefore makes sense to take the limit $t_{\mathrm{in,out}} \to \mp\infty$. The limit of the time evolution operator for infinite times is called the S-matrix

$$
\begin{aligned}
S &= \lim_{t_{\mathrm{in,out}} \to \mp\infty} \exp\!\big(iH_{\mathrm{as}}t_{\mathrm{out}}\big) \exp\!\big(iH(t_{\mathrm{in}} - t_{\mathrm{out}})\big) \exp\!\big(-iH_{\mathrm{as}}t_{\mathrm{in}}\big).\\
&= \lim_{t_{\mathrm{in,out}} \to \mp\infty} U(t_{\mathrm{out}}, t_{\mathrm{in}}) = U(+\infty, -\infty).
\end{aligned}
\qquad (10.22)
$$

It transforms in states to out states

$$|\mathrm{out}\rangle = S|\mathrm{in}\rangle. \qquad (10.23)$$

Note that the in and out Hilbert spaces are isomorphic.[6] This allows us to compare states between the two. To compute matrix elements of the S-matrix, prepare definite in and out states[7] using the creation and annihilation operators $a^\dagger, a$

$$
\begin{aligned}
|\mathrm{in}\rangle &= |p_1, \ldots, p_m\rangle := a^\dagger(\vec{p}_1) \ldots a^\dagger(\vec{p}_m)|0\rangle,\\
\langle\mathrm{out}| &= \langle q_1, \ldots, q_n| := \langle 0|a(\vec{q}_1) \ldots a(\vec{q}_n).
\end{aligned}
\qquad (10.24)
$$

Conventionally, scattering amplitudes M are defined as the matrix elements of $S - 1$ with the overall momentum-conserving delta function stripped off

$$\langle\mathrm{out}|(S - 1)|\mathrm{in}\rangle = (2\pi)^4 \delta^4(P_{\mathrm{in}} - P_{\mathrm{out}})iM(p_1, \ldots p_m; q_1, \ldots, q_n). \qquad (10.25)$$

The combination $S - 1$ is particularly useful for $2 \to n$ scattering processes: It removes all direct connections between the in and out states as well as all other disconnected contributions.[8]

Properties of the S-Matrix.   The S-matrix has a number of useful properties, let us list a few relevant ones.

First of all, the S-matrix is trivial for the ground state and for single-particle states

$$S|0\rangle = |0\rangle, \qquad S|\vec{p}\rangle = |\vec{p}\rangle. \qquad (10.26)$$

---

[6]It is natural to assume that outgoing particles of some scattering process can be used as ingoing particles of another scattering process. Therefore the in and out spaces must be isomorphic.

[7]These in and out states are not to be related by $|\mathrm{out}\rangle = S|\mathrm{in}\rangle$.

[8]When one of the ingoing particles does not participate in the scattering, the S-matrix must act trivially on the other. For general $m \to n$ scattering, the matrix elements indeed contain direct connections and disconnected contributions.

This follows from the definition of the asymptotic Hamiltonian to strictly emulate the action of the interacting Hamiltonian on these states.

The S-matrix is a unitary operator

$$S^\dagger = S^{-1}. \tag{10.27}$$

This property follows from the definition. It reflects the fact that probabilities are conserved across scattering processes.

The S-matrix is also Poincaré invariant

$$U(\omega, a) S U(\omega, a)^{-1} = S. \tag{10.28}$$

10.3 Time-Ordered Correlators

When we expressed the time-evolution operator in the interaction picture, we realised that time-ordered correlation functions $\langle \phi(x_1) \ldots \phi(x_n) \rangle$ are very natural objects. The S-matrix is defined as the time evolution operator for the interaction picture in terms of asymptotic states. Lehmann, Symanzik and Zimmermann derived a relationship between the S-matrix elements and time-ordered expectation values.

Asymptotic States. First we need to understand how to represent particle creation and annihilation operators $a^\dagger, a$ in terms of the field $\phi(x)$. Above we have expanded the field $\phi(x)$ as

$$\phi(x) = \sqrt{Z} \int \frac{d^3\vec{p}}{(2\pi)^3 \, 2e(\vec{p})} \left(e^{-ip \cdot x} a(\vec{p}) + e^{ip \cdot x} a^\dagger(\vec{p}) \right) + \ldots . \tag{10.29}$$

The omitted terms represent the contributions from multi-particle states and operators which annihilate the vacuum.

Previously we were able to isolate $a^\dagger(\vec{p})$ from a time slice of the free field $\phi_0(x)$ as

$$a^\dagger(\vec{p}) = \int d^3\vec{x} \, e^{-ip \cdot x} \left(e(\vec{p}) \phi_0(x) - i\dot{\phi}_0(x) \right). \tag{10.30}$$

This was easy because there are only two modes with $e = \pm e(\vec{p})$ in the free field ϕ_0. The linear combination of ϕ and $\dot{\phi}$ selects the correct one.

The interacting field, however, in general carries many other modes whose precise nature we do not understand a priori. To select the modes corresponding to $a^\dagger$ and a we need to drive the field $\phi(x)$ for a sufficiently long time with a frequency that is in resonance with the relevant modes. Let us sketch the construction for a single oscillator $f(t) = c e^{i\omega t}$ with resonance frequency ω

$$F(e) = \int_{t_1}^{t_2} dt \, e^{-iet} f(t) = \frac{ic}{e - \omega} \left(e^{-i(e-\omega)t_2} - e^{-i(e-\omega)t_1} \right) \tag{10.31}$$

The longer the time, the stronger will be the amplitude at $e = \omega$. At infinite time the function $F(e)$ develops a pole at e, so we set $t_1 = -\infty$

$$F(e) = \int_{-\infty}^{t_2} dt\, e^{-iet} f(t) = \frac{ice^{-i(e-\omega)t_2}}{e - \omega} = \frac{ic}{e - \omega} + \text{finite}. \qquad (10.32)$$

Here we have discarded the term that keeps oscillating at asymptotic times.[9] What remains is an isolated pole at $e = \omega$ whose residue is proportional to the amplitude c. The residue is in fact independent of the time t_2 where the driving stops.

Applied to the field $\phi(x)$ we find

$$\int_{-\infty}^{t_2} dt \int d^3\vec{x}\, e^{-ip\cdot x} \phi(x) \qquad (10.33)$$

$$= \frac{i\sqrt{Z}}{p^2 + m^2} \left(\theta(-e)a_{\text{in}}(-\vec{p}) - \theta(e)a_{\text{in}}^\dagger(\vec{p}) \right) + \dots. \qquad (10.34)$$

What remains are isolated poles at $e = \pm e(\vec{p})$ whose residues are creation and annihilation operators for ingoing asymptotic particles. The remaining terms are either finite or irrelevant when creating well-separated wave packets. We decided to shift the ingoing Fock space back to the time $t = 0$ using the free asymptotic Hamiltonian. Therefore we conjugate the creation and annihilation operators by the appropriate time evolution operator

$$a_{\text{in}}(\vec{p}) = U(0, -\infty)a(\vec{p})U(-\infty, 0). \qquad (10.35)$$

We note:

- The residues of the pole $1/(p^2 + m^2)$ isolate the creation and annihilation operators.[10]
- The residues at positive and negative energies correspond to creation and annihilation operators, respectively.
- The residues do not depend on the final time t_2.
- Bound state particles correspond to similar poles at different energies.

A similar expression with opposite sign is obtained for driving the field into the distant future

$$\int_{t_1}^{\infty} dt \int d^3\vec{x}\, e^{-ip\cdot x} \phi(x) \qquad (10.36)$$

$$= -\frac{i\sqrt{Z}}{p^2 + m^2} \left(\theta(-e)a_{\text{out}}(-\vec{p}) - \theta(e)a_{\text{out}}^\dagger(\vec{p}) \right) + \dots. \qquad (10.37)$$

Here we identify

$$a_{\text{out}}(\vec{p}) = U(0, +\infty)a(\vec{p})U(+\infty, 0). \qquad (10.38)$$

---

[9]As usual, one could formally dampen this term by introducing some small imaginary part. This may be an approximation, but even in practice, one can *never* isolate a resonance perfectly.

[10]The can be further operators consisting of several creation and annihilation operators.

LSZ Reduction. We want to express the elements of the S-matrix in terms of time-ordered correlation functions in momentum space. Let us start with the time-ordered expectation value

$$F_{m,n}(p,q) = \int \prod_{k=1}^{m} \left(d^4x_k \, e^{-ip_k \cdot x_k} \right) \prod_{k=1}^{n} \left(d^4y_k \, e^{iq_k \cdot y_k} \right)$$
$$\langle 0 | T\big(\phi(x_1)\dots\phi(x_n)\phi(y_1)\dots\phi(y_m)\big) | 0 \rangle. \tag{10.39}$$

Consider just the integral of the quantum operator over one of the x_k

$$X = \int d^4x \, e^{-ip \cdot x} T\big(\phi(x)Y\big). \tag{10.40}$$

Now split up the time integral into three regions at the times $t_{\min}$ and $t_{\max}$ representing the minimal and maximal times within the operator Y

$$X = \int_{-\infty}^{t_{\min}} dt \int d^3\vec{x} \, e^{-ip \cdot x} T(Y)\phi(x)$$
$$+ \int_{t_{\min}}^{t_{\max}} dt \int d^3\vec{x} \, e^{-ip \cdot x} T\big(\phi(x)Y\big)$$
$$+ \int_{t_{\max}}^{+\infty} dt \int d^3\vec{x} \, e^{-ip \cdot x} \phi(x)T(Y). \tag{10.41}$$

According to the results of the above consideration of resonances, the two integrals extending to $t = \pm\infty$ produce a pole when the momentum is on shell, $p^2 = -m^2$. Conversely, the middle integral is finite and therefore does not produce a pole. We can express the residue of the pole using creation operators of in and out particles

$$X \simeq \frac{-i\sqrt{Z}}{p^2 + m^2} \left(T(Y)a_{\text{in}}^\dagger(\vec{p}) - a_{\text{out}}^\dagger(\vec{p})T(Y) \right), \tag{10.42}$$

where we discard finite contributions at $p^2 = -m^2$. Performing this step for all ingoing particles yields

$$F_{m,n} \simeq \prod_{k=1}^{m} \left(\frac{-i\sqrt{Z}}{p_k^2 + m^2} \right) \int \prod_{k=1}^{n} \left(d^4y_k \, e^{iq_k \cdot y_k} \right)$$
$$\langle 0 | T\big(\phi(y_1)\dots\phi(y_m)\big)a_{\text{in}}^\dagger(\vec{p}_1)\dots a_{\text{in}}^\dagger(\vec{p}_m) | 0 \rangle$$
$$= \prod_{k=1}^{m} \left(\frac{-i\sqrt{Z}}{p_k^2 + m^2} \right) \int \prod_{k=1}^{n} \left(d^4y_k \, e^{iq_k \cdot y_k} \right)$$
$$\langle 0 | T\big(\phi(y_1)\dots\phi(y_m)\big)U(0,-\infty)a^\dagger(\vec{p}_1)\dots a^\dagger(\vec{p}_m) | 0 \rangle. \tag{10.43}$$

Note that all outgoing creation operators $a_{\text{out}}^\dagger$ directly annihilate the vacuum $\langle 0 |$.

Now we perform equivalent steps for the outgoing particles. We use a similar

10.8

relation as above dressed by factors of $U(+\infty, 0)$ and $U(0, -\infty)$

$$X = \int d^4 y \, e^{iq \cdot y} U(+\infty, 0) \mathrm{T}\big(\phi(y)Y\big) U(0, -\infty). \tag{10.44}$$

$$\simeq \frac{-i\sqrt{Z}}{q^2 + m^2} \Big(U(+\infty, 0)\mathrm{T}(Y)a_{\mathrm{in}}(\vec{q})U(0, -\infty) \tag{10.45}$$

$$- U(+\infty, 0)a_{\mathrm{out}}(\vec{q})\mathrm{T}(Y)U(0, -\infty)\Big), \tag{10.46}$$

$$\simeq \frac{-i\sqrt{Z}}{q^2 + m^2} [U(+\infty, 0)\mathrm{T}(Y)U(0, -\infty), a(\vec{q})]. \tag{10.47}$$

For each particle this yields one commutator of the remaining fields $U(+\infty, 0)\mathrm{T}(Y)U(0, -\infty)$ with an annihilation operator. After performing this step for all the outgoing particles, we are left with the S-matrix

$$U(+\infty, 0)\mathrm{T}(1)U(0, -\infty) = S. \tag{10.48}$$

Altogether we find that the residue of $F_{m,n}$ is given by an element of the S-matrix

$$F_{m,n} \simeq \prod_{k=1}^{m} \left(\frac{-i\sqrt{Z}}{p_k^2 + m^2} \right) \prod_{k=1}^{n} \left(\frac{-i\sqrt{Z}}{q_k^2 + m^2} \right)$$
$$\langle 0|[a(\vec{q}_1), \dots [a(\vec{q}_n), S] \dots]a^\dagger(\vec{p}_1) \dots a^\dagger(\vec{p}_m)|0\rangle. \tag{10.49}$$

Here, the commutators make all the $a(\vec{q}_k)$ connect only to the S-matrix. Now there is nothing else left, and therefore also all $a^\dagger(\vec{p}_k)$ must connect to S.

10.4 S-Matrix Reconstruction

We have seen that time-ordered correlation functions have poles when the external fields are on the mass shell of asymptotic particles. The residue of these poles is given by the corresponding element of the scattering matrix.

We can therefore fully reconstruct the S-matrix from time-ordered correlation functions.

Two-Point Correlator. In the construction of the S-matrix, the two-point correlation function takes a special role. First, consider the above residue formula for two legs

$$F_{1,1} \simeq \frac{-i\sqrt{Z}}{p^2 + m^2} \frac{-i\sqrt{Z}}{q^2 + m^2} \langle 0|a(\vec{q})(S - 1)a^\dagger(\vec{p})|0\rangle. \tag{10.50}$$

Momentum conservation implies $p = q$, hence the residue of a double pole at $p^2 = -m^2$ is given by $\langle 0|a(\vec{q})(S - 1)a^\dagger(\vec{p})|0\rangle$. However, the S-matrix should act as the identity on single-particle states. We conclude that there is no double pole in $F_{1,1}$ at $p^2 = -m^2$. There is no reason to expect a double pole in the first place, therefore the above residue statement is empty for $m = n = 1$.

10.9

There is nevertheless a single pole at $p^2 = -m^2$ as can be shown using the spectral representation of the time-ordered two-point function

$$F_2(x - y) = \langle \phi(x)\phi(y) \rangle := \langle 0|T(\phi(x)\phi(y))|0\rangle. \qquad (10.51)$$

Using the spectral function $\rho(s)$ of the interacting field $\phi(x)$, it can be written in terms of the free Feynman propagator of mass $\sqrt{s}$

$$F_2(x - y) = -i \int_0^\infty \frac{ds}{2\pi} \rho(s) G_\mathrm{F}(s; x - y). \qquad (10.52)$$

Most importantly, its momentum space representation

$$F_2(p) = -i \int_0^\infty \frac{ds}{2\pi} \frac{\rho(s)}{p^2 + s - i\epsilon} = \frac{-iZ}{p^2 + m^2 - i\epsilon} + \dots \qquad (10.53)$$

contains the parameters of the asymptotic particle: The function $F_2(p)$ has an isolated pole at the physical mass m, and its residue is the wave function renormalisation factor Z.

Now we can nicely expand F_2 in terms of Feynman diagrams with two external legs and thus determine m and Z.

Amputation. The residue formula for the time-ordered correlation functions can be inverted to a complete expression for the S-matrix

$$S = 1 + \sum_{m,n=2}^\infty \int \prod_{k=1}^n \frac{d^3\vec{q}_k \, a^\dagger(\vec{q}_k)}{(2\pi)^3 \, 2e(\vec{q})} \prod_{k=1}^m \frac{d^3\vec{p}_k \, a(\vec{p}_k)}{(2\pi)^3 \, 2e(\vec{p})}$$
$$\cdot \left(\frac{F_{m,n}(p, q)}{m! \, n!} \prod_{k=1}^m \frac{p_k^2 + m^2}{-i\sqrt{Z}} \prod_{k=1}^n \frac{q_k^2 + m^2}{-i\sqrt{Z}} \right).$$

Importantly, the poles and zeros of the latter term must be combined before the momenta are set on shell $p_k^2 = q_k^2 = -m^2$. The construction of this expression ensures that

- the vacuum does not scatter, $S|0\rangle = |0\rangle$,
- single-particle states do not scatter, $S|p\rangle = |p\rangle$,
- for more two or more particles, the residue of $F_{m,n}$ is reproduced according to the above formula.

It is now convenient to replace each factor $(p_k^2 + m^2)$ by the inverse of the corresponding two-point function in the construction of the S-matrix

$$S = 1 + \sum_{m,n=2}^\infty \int \prod_{k=1}^n \frac{d^3\vec{q}_k \, a^\dagger(\vec{q}_k)}{(2\pi)^3 \, 2e(\vec{q})} \prod_{k=1}^m \frac{d^3\vec{p}_k \, a(\vec{p}_k)}{(2\pi)^3 \, 2e(\vec{p})}$$
$$\cdot \left(\frac{F_{m,n}(p, q)}{m! \, n!} \prod_{k=1}^m \frac{\sqrt{Z}}{F_2(p_k)} \prod_{k=1}^n \frac{\sqrt{Z}}{F_2(q_k)} \right).$$

This formula has a useful interpretation in terms of Feynman graphs for $F_{m,n}$.

$$(10.54)$$

In the second representation we have cut the graph into a smaller $(m+n)$-function and $m+n$ 2-point functions according to the rules:

- Each 2-point function connects an external leg to the $(m+n)$-function at the core.
- Each 2-point function is maximal.
- The Feynman propagator that connects the 2-point function to the core is attributed to the 2-point function.

Essentially one chops each leg of the graph as much as possible. Such a graph is called *amputated.*

Now it is clear that each 2-point fragment of the graph is a Feynman graph for the two-point function F_2. Moreover all these graphs have natural relative weights. The sum of all Feynman graphs contributing to $F_{m,n}$ therefore contains the sum of all graphs contributing to F_2 separately for each leg

$$(10.55)$$

What remains is a sum over all amputated Feynman graphs at the core. This expression separates cleanly into factors because all the weights are naturally defined

$$F_{m,n}(p,q) = \tilde{F}_{m,n}(p,q) \prod_{k=1}^{m} F_2(p_k) \prod_{k=1}^{n} F_2(q_k). \qquad (10.56)$$

The function $\tilde{F}_{m,n}$ therefore is precisely what is needed for reconstruction of the S-matrix

$$S = 1 + \sum_{m,n=2}^{\infty} \int \prod_{k=1}^{n} \frac{d^3\vec{q}_k \, a^\dagger(\vec{q}_k)}{(2\pi)^3 \, 2e(\vec{q})} \prod_{k=1}^{m} \frac{d^3\vec{p}_k \, a(\vec{p}_k)}{(2\pi)^3 \, 2e(\vec{p})} \frac{\sqrt{Z}^{m+n}}{m! \, n!} \, \tilde{F}_{m,n}.$$

In other words, the elements of the S-matrix are determined precisely by the sum of amputated Feynman graphs multiplied by $\sqrt{Z}$ for each external leg.

10.11

In General. The general picture is as follows: Poles in the time-ordered two-point function $F_2(p)$ indicate stable asymptotic particle states.[11] [12]

- These may be deformations of the poles in the free theory.
- They may as well be poles corresponding to bound states.
- Also poles for correlators of composite fields are permissible.

The location $p^2 = -m^2$ of the pole defines the mass m of the particle. Time-ordered multi-point correlation functions have poles at these locations. Their overall residue yields the corresponding element of the S-matrix. Some comments:

- It is clear that all the external legs of the S-matrix must be exactly on shell.
- Note that in this picture of the S-matrix, crossing symmetry follows from crossing symmetry of time-ordered correlators.
- The S-matrix is completely determined in terms of time-ordered correlation functions. No reference is made to the original formulation of the QFT, e.g. the Lagrangian. This fact will be crucial when we go to higher perturbative orders where Feynman diagrams have internal loops.

Feynman Rules. Let us summarise the Feynman rules for elements of the S-matrix in ϕ^4 theory

$$\langle q_1, \ldots, q_n | S | p_1, \ldots, p_m \rangle. \tag{10.57}$$

The matrix element is given by the sum of all graphs with certain properties. The properties are similar to the properties of Feynman graphs for correlation functions in momentum space, but mainly the external legs are handled differently. Let us state the modified and additional rules:

- The graph has m ingoing and n outgoing external lines labelled by momenta p_k and q_k, respectively.
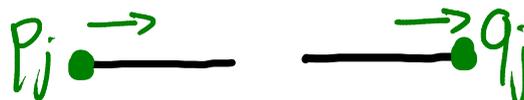
 (10.58)

- The external momenta must be on the mass shell, $p_k^2 = q_k^2 = -m^2$, and must have positive energy.
- Cutting the graph at any internal line must not split off a graph with two

---

[11] When a field has several components, the notion of pole is more subtle in the sense that the residue of a pole is typically a matrix of non-maximal rank, e.g. $p \cdot \gamma + m$ for spinor fields. In this case only the vectors which are not projected out correspond to asymptotic particles.

[12] In practice one may not be able to distinguish an exact pole from a very narrow resonance. One might consider such resonances at the same level as stable external particles and allow them as legs of the S-matrix. Such an S-matrix would not rest on rigorous assumptions and therefore not all theorems apply in a strict sense. In this regard, one should remember that in quantum physics one has to make some separation of scales into the microscopic quantum regime and the regime of macroscopic classical objects. Alternatively, resonances can be viewed as asymptotic particles with a complex mass parameter.

external lines (amputated graph).
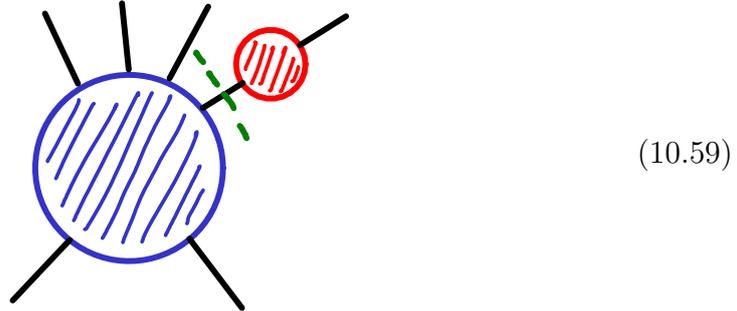
$$(10.59)$$

The Feynman rules for evaluating a graph are the same as for correlation functions in momentum space except:

- For each external line write a factor of $\sqrt{Z}$ instead of a Feynman propagator $-i/(p_j^2 + m^2 - i\epsilon)$.

$$\rightarrow \sqrt{Z}. \qquad (10.60)$$

- Any external line which directly connects an ingoing to an outgoing particle contributes a factor of $\langle q_k | p_l \rangle = 2e(\vec{p}_l)(2\pi)^3 \delta^3(\vec{p}_l - \vec{q}_k)$. This line simply bypasses the S-matrix.[13]

$$\rightarrow 2e(\vec{p}_j)\,(2\pi)^3 \delta^3(\vec{p}_j - \vec{q}_k). \qquad (10.61)$$

For Quantum Electrodynamics the Feynman rules for scattering matrix elements also has to be adjusted w.r.t. the Feynman rules in momentum space, namely:

- For each external spinor line, write a factor of $\sqrt{Z_\psi}$ along with $u_\alpha(\vec{q})$, $\bar{u}_\alpha(\vec{p})$, $v_\alpha(\vec{p})$ or $\bar{v}_\alpha(\vec{q})$ depending on whether the particle is in- or outgoing and whether it is an electron or a positron.

$$\rightarrow \sqrt{Z_\psi}\, v_{\alpha_j}(\vec{p}_j), \qquad \rightarrow \sqrt{Z_\psi}\, \bar{u}_{\alpha_j}(\vec{p}_j),$$

$$\rightarrow \sqrt{Z_\psi}\, \bar{v}_{\alpha_j}(\vec{q}_j), \qquad \rightarrow \sqrt{Z_\psi}\, u_{\alpha_j}(\vec{q}_j) \qquad (10.62)$$

- For external photon lines, write a factor of $\sqrt{Z_A}$ along with a normalised transverse polarisation vector $e_\mu(\vec{p})$.

$$\rightarrow \sqrt{Z_A}\, \varepsilon_{\alpha_j}(\vec{p}_j), \qquad \rightarrow \sqrt{Z_A}\, \varepsilon^*_{\alpha_j}(\vec{q}_j) \qquad (10.63)$$

---

[13]Such contributions do not directly correspond to the identity within S, i.e. they are present in $S - 1$, but only for at least 3 ingoing particles.

10.5 Unitarity

The S-matrix is a unitary operator

$$S^\dagger = S^{-1}. \tag{10.64}$$

This is an essential feature of any physical QFT. However, when deriving the S-matrix from time-ordered correlators by means of the LSZ reduction, unitarity is not evident at all. Therefore we can use the property to derive some non-trivial relations between elements of the S-matrix.

Optical Theorem. Commonly, an identity operator is removed from the S-matrix as

$$S = 1 + iT. \tag{10.65}$$

This split is useful because for small coupling T is small. Moreover, the identity in S is never seen in LSZ reduction.

Unitarity $SS^\dagger = 1$ for the operator T is then written as the optical theorem

$$2 \operatorname{Im} T = -iT + iT^\dagger = TT^\dagger = T^\dagger T. \tag{10.66}$$

It relates the imaginary part of T to its absolute square. The latter is a quantity we have already encountered: In the form of matrix elements it appears in the scattering cross section. It allows to determine the total cross section of some process in terms of the imaginary part of a matrix element.[14] Alternatively, the imaginary part of T can be obtained as a total cross section,[15] The remaining real part of T can be reconstructed from arguments of complex analyticity.

A graphical representation of the optical theorem is as follows

$$2 \operatorname{Im} \left(\ldots T \ldots \right) = \sum_{l=2}^{\infty} \prod_{j=1}^{l} \int \frac{d^3 \vec{k}_j}{(2\pi)^3 \, 2e(\vec{k}_j)} \sum_{\text{pol}} \left(\ldots \right). \tag{10.67}$$

The optical theorem implies that one has to integrate and sum over all allowed degrees of freedom for these lines which connect T to $T^\dagger$. This is similar as for internal lines within T and $T^\dagger$ with one important distinction: The cut lines originate from contracting two operators a and $a^\dagger$ inside T and $T^\dagger$, respectively,

$$[a(\vec{p}), a^\dagger(\vec{q})] = 2e(\vec{p}) \, (2\pi)^3 \delta^3(\vec{p} - \vec{q}). \tag{10.68}$$

---

[14]In this matrix element one would choose the ingoing and outgoing momenta to be the same. Evidently, this requires to split off the momentum-conserving delta function first.

[15]In fact one needs a generalisation of the total cross section where the ingoing particles of T are chosen independently of the outgoing particles of $T^\dagger$.

Therefore the momenta associated to these lines must be on shell, $p^2 = -m^2$, with directed flow of energy p_0 from T towards $T^\dagger$. Conversely, the internal lines are integrated over all off-shell momenta.

Tree Level.   It is instructive to discuss the optical theorem at tree level. At first sight one might think that tree-level contributions to T are manifestly real because they are rational functions of the momenta and masses with real coefficients.[16] Although the $i\epsilon$ prescription for Feynman propagators appears negligible, it does have a considerable impact on the imaginary part

$$\frac{1}{p^2 + m^2 - i\epsilon} = \frac{1}{p^2 + m^2} + i\pi\delta(p^2 + m^2). \tag{10.69}$$

Now in the conjugate S-matrix $T^\dagger$ all Feynman propagators are conjugated

$$G_{\mathrm{F}}^*(p) = \frac{1}{p^2 + m^2 + i\epsilon} \neq G_{\mathrm{F}}(p). \tag{10.70}$$

When computing the imaginary part of T one therefore frequently encounters the difference

$$\frac{1}{p^2 + m^2 - i\epsilon} - \frac{1}{p^2 + m^2 + i\epsilon} = 2\pi i\delta(p^2 + m^2). \tag{10.71}$$

This identity replaces the Feynman propagator for an internal line by an on-shell correlator for a cut line connecting T and $T^\dagger$. The restriction to positive energies on the cut is a more subtle issue. It is resolved by the fact that in the sum over all possible cuts each line appears twice, once for every direction of energy flow.

With these remarks one can show that the optical theorem holds at tree level.[17] Here we showed that at tree level T is has an imaginary part concentrated at isolated momentum configurations. However, the optical theorem is most frequently applied at loop level where T is generically complex.

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[16]The various prefactors of i for propagators and interaction vertices conspire to cancel out.

[17]Here it is crucial to also take the disconnected contributions to T into account.

11 Loop Corrections

Finally, we will discuss some basic loop effects and how to deal with the divergent loop integrals we shall encounter.

We use a scalar theory with a quartic potential

$$\mathcal{L} = -\tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}\mu^2\phi^2 - \tfrac{1}{6}\kappa\mu\phi^3 - \tfrac{1}{24}\kappa^2\lambda\phi^4. \tag{11.1}$$

The mass is denoted by μ and there are two dimensionless constants κ and λ. We assume κ to be small and λ to be a freely tunable parameter.

11.1 Self Energy

First, we consider the time-ordered two-point correlator in order to isolate the pole associated to asymptotic particles. In momentum space we know by means of momentum conservation

$$F_2(p, q) = -i(2\pi)^4\delta^4(p + q)M_2(p). \tag{11.2}$$

Leading Orders. Let us evaluate the first few orders of the function M_2 from Feynman diagrams.
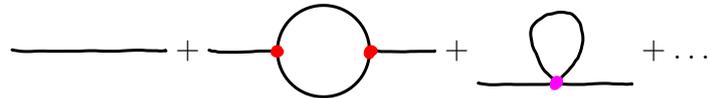
$$\tag{11.3}$$

Obviously the leading contribution is the isolated Feynman propagator

$$M_2^{(0)}(p) = \frac{1}{p^2 + \mu^2 - i\epsilon}. \tag{11.4}$$

The first correction involves one loop in the diagram. The tadpole diagram can be safely ignored.[1] The bubble diagram amounts to the following loop integral[2]

$$M_2^{(2)}(p) = \frac{i(-i\mu\kappa)^2(-i)^4}{(p^2 + \mu^2 - i\epsilon)^2}\, iI(-p^2)\,,$$

$$I(-p^2) = \frac{1}{2}\int\frac{-id^4\ell}{(2\pi)^4}\,\frac{1}{\ell^2 + \mu^2 - i\epsilon}\,\frac{1}{(p - \ell)^2 + \mu^2 - i\epsilon}\,. \tag{11.5}$$

The resulting quadruple integral is difficult to perform, in particular due to two different denominators. We will postpone the evaluation, and discuss the implications for a generic function $I(-p^2)$.

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[1] We shall see later on how this can be achieved in practice.

[2] The resulting integral is Lorentz invariant and can therefore be written as a function of p^2 (or $-p^2$).

Mass Shift.   Altogether we obtain for the two-point function

$$M_2(p) = \frac{1}{p^2 + \mu^2 - i\epsilon} + \frac{\mu^2 \kappa^2 I(-p^2)}{(p^2 + \mu^2 - i\epsilon)^2} + \dots \,. \tag{11.6}$$

Now this expression appears to have a double pole at $p^2 = -\mu^2$. From our earlier discussion we know that this should not happen; there should only be single poles in the two-point function.

Taking a peek at higher loop orders, we find, among others, a sequence of iterated loop integrals terms.

$$\tag{11.7}$$

All of these we can express easily in the form of a geometric series

$$M_2(p) = \frac{1}{p^2 + \mu^2 - i\epsilon} \sum_{k=0}^{\infty} \left(\frac{\mu^2 \kappa^2 I(-p^2)}{p^2 + \mu^2 - i\epsilon} \right)^k + \dots \,. \tag{11.8}$$

Pushing convergence questions aside, we sum the series

$$M_2(p) = \frac{1}{p^2 + \mu^2 - \mu^2 \kappa^2 I(-p^2) - i\epsilon} + \dots \,. \tag{11.9}$$

In other words we have moved the correction term to the denominator of the propagator.

Why should we include precisely these higher-order terms into our first order correction? There are several reasons:

- There is nothing wrong with it. We just have to make sure to eventually count every higher-order diagram with the correct weight.
- We avoid unwanted higher poles in the two-point function. The diagrams contributing to the denominator are one-particle irreducible. They cannot be cut into two parts by cutting a single internal line. Such diagrams are not expected to produce additional poles at the mass shell.
- The inverse of the two-point function at leading order is directly related to the action. It therefore appears somewhat more natural to expand the inverse $1/M_2(p)$ rather than the original function $M_2(p)$. Our correction terms are simply the first correction to $1/M_2(p)$.
- A mass counterterm (to be discussed below) yields the same result.
- In QFT2 we will introduce a useful functional that includes $1/M_2(p)$.

In this form we generically expect only single poles. Particularly, the pole which was originally at $p^2 + \mu^2 = 0$ may now have shifted to a new location $p^2 + m^2 = 0$. Assuming that this is the case, we can determine the new mass and also the residue at the pole. For the mass, we should solve the equation

$$-m^2 + \mu^2 - \mu^2 \kappa^2 I(m^2) + \dots = 0. \tag{11.10}$$

The assumption of the perturbative treatment is that κ is small and that m is approximated well by μ. Hence we can replace $I(m^2)$ by $I(\mu^2)$ and therefore the new mass to leading order reads

$$m^2 = \mu^2 - \mu^2 \kappa^2 I(\mu^2) + \dots \,. \tag{11.11}$$

11.2

How about the residue? The expansion of the denominator at $p^2 + m^2 = 0$ reads

$$(p^2 + m^2)\left(1 + \mu^2\kappa^2 I'(m^2)\right) + \dots. \tag{11.12}$$

Altogether the pole of the two-point function corresponding to the asymptotic particle takes the form

$$M_2(p) = \frac{Z}{p^2 + m^2 - i\epsilon} + \dots \tag{11.13}$$

with the field strength renormalisation

$$Z = 1 - \mu^2\kappa^2 I'(\mu^2) + \dots. \tag{11.14}$$

Spectral Function. We can now extract the spectral function $\rho(s)$ from $M_2(p)$

$$M_2(p) = \int_0^\infty \frac{ds}{2\pi}\frac{\rho(s)}{p^2 + s - i\epsilon}. \tag{11.15}$$

To that end, it is most convenient to consider the imaginary part originating from the $i\epsilon$ prescription

$$\frac{1}{x - i\epsilon} = \frac{1}{x} + i\pi\delta(x). \tag{11.16}$$

We thus find that the imaginary part of M_2 is directly related to the spectral function up to a factor of 2

$$\rho(-p^2) = 2\,\mathrm{Im}\,M_2(p). \tag{11.17}$$

We can evaluate the imaginary part of M_2 using the derivative of the above identity for the $i\epsilon$ prescription

$$\frac{1}{(x - i\epsilon)^2} = \frac{1}{x^2} - i\pi\delta'(x). \tag{11.18}$$

We then find the spectral function

$$\begin{aligned}
\rho(s) &= 2\pi\delta(-s + \mu^2) - 2\pi\mu^2\kappa^2\,\mathrm{Re}\,I(s)\delta'(-s + \mu^2) \\
&\quad + \frac{2\mu^2\kappa^2\,\mathrm{Im}\,I(s)}{(-s + \mu^2)^2} + \dots \\
&= 2\pi\delta\left(s - \mu^2 + \mu^2\kappa^2\,\mathrm{Re}\,I(\mu^2)\right)\left(1 - \mu^2\kappa^2\,\mathrm{Re}\,I'(\mu^2)\right) \\
&\quad + \frac{2\mu^2\kappa^2\,\mathrm{Im}\,I(s)}{(s - \mu^2)^2} + \dots \\
&= 2\pi Z\delta\left(s - m^2\right) + \frac{2\mu^2\kappa^2\,\mathrm{Im}\,I(s)}{(s - \mu^2)^2} + \dots,
\end{aligned} \tag{11.19}$$

which matches precisely with our expectations.

$$\tag{11.20}$$

11.3
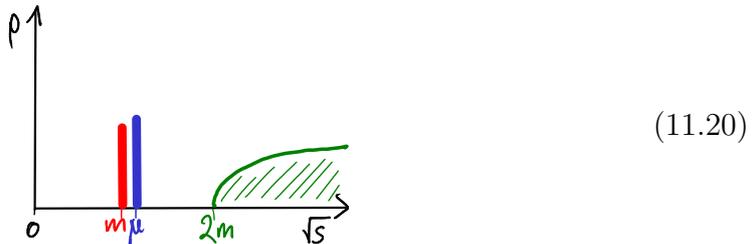
11.2 Loop Integral

We will now turn back to the loop integral $I(-p^2)$ and evaluate it.

Combining Denominators. A standard trick to proceed due to Feynman is to combine the denominators by virtue of a new integral. For two propagators we use the integral[3]

$$\frac{1}{AB} = \int_0^1 \frac{dz}{(zA + \bar{z}B)^2}, \qquad \bar{z} := 1 - z. \tag{11.21}$$

An alternative is the trick used by Schwinger to convert each numerator to an exponent where they automatically sum up

$$\frac{1}{A} = \int_0^\infty dz \, e^{-Az}. \tag{11.22}$$

The loop integral then takes the form with a single squared denominator[4]

$$I(-p^2) = \int_0^1 dz \int \frac{-id^4\ell}{32\pi^4} \frac{1}{\left(z\ell^2 + \bar{z}(p - \ell)^2 + \mu^2 - i\epsilon\right)^2}. \tag{11.23}$$

Now the denominator has quadratic (ℓ^2), linear ($\ell{\cdot}p$) and constant terms. We can remove the linear term and thus simplify the integral by a shift $\ell \to \ell + \bar{z}p$ of the integration variable[5]

$$I(-p^2) = \int_0^1 dz \int \frac{-id^4\ell}{32\pi^4} \frac{1}{\left(\ell^2 + z\bar{z}p^2 + \mu^2 - i\epsilon\right)^2}. \tag{11.24}$$

Momentum Integrals. Performing the momentum integrals is not very difficult. Let us start with the integral over the energy component. The integrand has double and single poles at

$$\ell_0 = \pm\sqrt{\vec{\ell}\,^2 + z\bar{z}p^2 + \mu^2 - i\epsilon}. \tag{11.25}$$

The integration contour is along the real axis and passes right between the two poles. It decays sufficiently fast at $|l_0| \to \infty$ so that we can close the contour by a large semicircle in the upper or lower half of the complex plane. Either of the single poles contributes the same residue

$$I(-p^2) = \int_0^1 dz \int \frac{d^3\vec{\ell}}{64\pi^3} \frac{1}{\left(\vec{\ell}\,^2 + z\bar{z}p^2 + \mu^2 - i\epsilon\right)^{3/2}}. \tag{11.26}$$

---

[3]There are similar formulas for more than two denominators and for higher powers.

[4]The advantage of this expression is that the new integrand is spherically symmetric for fixed value of z which simplifies integration drastically. The centre of the sphere, however, varies with z.

[5]For an infinite integration domain shifting the integration variable does not change the integral as long as it is convergent. For divergent integrals this point is subtle.

We notice that the integral merely reduces the exponent of the denominator by $1/2$ and multiplies by a suitable overall factor. The remaining three spatial momentum integrals yield a very similar result, each of them reducing the exponent by $1/2$. The last integral in fact is logarithmically divergent for large momenta. We have to cut it off at some bound $|\vec{l}| \simeq \Lambda_{\mathrm{cut}}$, and we obtain

$$I(-p^2) = -\frac{1}{32\pi^2} \int_0^1 dz \log \frac{z\bar{z}p^2 + \mu^2 - i\epsilon}{\Lambda_{\mathrm{cut}}^2}. \tag{11.27}$$

For a large UV cutoff Λ_{cut} the integral diverges logarithmically.

Wick Rotation. Another trick which is commonly used is to rotate the integration contour for the energy ℓ_0 from the real axis to the imaginary axis

$$\ell_0 = i\ell_{\mathrm{E},4}. \tag{11.28}$$

This Wick rotation is permissible since physical integrands are typically perfectly analytic in the first and third quadrants of the complex plane. For instance, poles of Feynman propagators are located slightly below the positive real axis in quadrant four or slightly above the negative real axis in quadrant two.

$$(11.29)$$
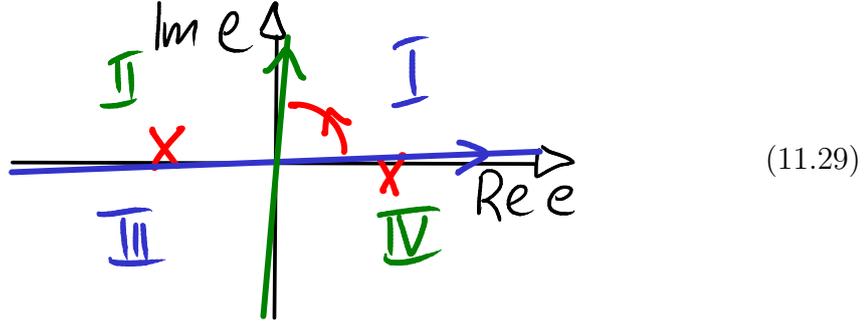
By means of the residue theorem, the value of the integral does not change by the Wick rotation

$$\int d^4\ell\, F(\ell_0, \vec{\ell}) = \int i d^4\ell_{\mathrm{E}}\, F(i\ell_{\mathrm{E},4}, \vec{\ell}). \tag{11.30}$$

Applying this rotation to our loop integral we obtain an integral over 4-dimensional Euclidean space

$$I(-p^2) = \int_0^1 dz \int \frac{d^4\ell_{\mathrm{E}}}{32\pi^4} \frac{1}{\left(\ell_{\mathrm{E}}^2 + z\bar{z}p^2 + \mu^2 - i\epsilon\right)^2}. \tag{11.31}$$

Now the integrand depends only on $|\ell_{\mathrm{E}}|$, and we can use rotational symmetry to replace the integral over three spherical angles at fixed $|\ell_{\mathrm{E}}|$ by the volume of a three-sphere $2\pi^2|\ell_{\mathrm{E}}|^3$

$$\int d^4\ell_{\mathrm{E}} F(|\ell_{\mathrm{E}}|) = 2\pi^2 \int_0^\infty \ell_{\mathrm{E}}^3\, d\ell_{\mathrm{E}} F(\ell_{\mathrm{E}}). \tag{11.32}$$

11.5

For our integral this implies

$$I(-p^2) = \int_0^1 dz \int_0^\infty \frac{d\ell_{\mathrm{E}}}{16\pi^2} \frac{\ell_{\mathrm{E}}^3}{\left(\ell_{\mathrm{E}}^2 + z\bar{z}p^2 + \mu^2 - i\epsilon\right)^2} . \tag{11.33}$$

The integral over ℓ_{E} is divergent again an needs to be cut off at $|\ell_{\mathrm{E}}| \simeq \Lambda_{\mathrm{cut}}$. The result is compatible with the above expression up to some minor adjustment of the cutoff parameter Λ_{cut}.

Final Integral. Gladly the remaining integral over the Feynman parameter z can be performed for our simple integral yielding

$$I(-p^2) = -\frac{1}{16\pi^2} \sqrt{\frac{p^2 + 4\mu^2 - i\epsilon}{-p^2}} \arctan \sqrt{\frac{-p^2}{p^2 + 4\mu^2 - i\epsilon}}$$
$$- \frac{1}{32\pi^2} \log \frac{\mu^2}{\Lambda_{\mathrm{cut}}^2 e^2} . \tag{11.34}$$

The integral is manifestly real for $0 < -p^2 < 4\mu^2$. It is also real for $-p^2 < 0$. However, for $-p^2 > 4\mu^2$ it develops an imaginary part

$$\mathrm{Im}\, I(-p^2) = \frac{1}{32\pi} \sqrt{\frac{-p^2 - 4\mu^2}{-p^2}} . \tag{11.35}$$

It signals the opening of the two-particle creation channel at $-p^2 > 4\mu^2$.

Spectral Function. We can now write the spectral function for our model at next-to-leading order

$$\rho(s) = 2\pi Z \delta\left(s - m^2\right) + \frac{\mu^2 \kappa^2 \theta(s - 4\mu^2)}{16\pi(s - \mu^2)^2} \sqrt{\frac{s - 4\mu^2}{s}} + \dots . \tag{11.36}$$

The two terms correspond to the asymptotic particle and the two-particle continuum. Let us consider its normalisation, we find

$$\int \frac{ds}{2\pi} \rho(s) = Z + \kappa^2 \frac{2\sqrt{3}\pi - 9}{288\pi^2} + \dots \tag{11.37}$$

Since $2\sqrt{3}\pi > 9$ we see that the correction term due to the two-particle continuum is indeed small and positive. Reassuringly the field strength renormalisation precisely compensates for the correction term

$$Z = 1 - \mu^2 \kappa^2 I'(\mu^2) + \dots, \qquad I'(\mu^2) = \frac{2\sqrt{3}\pi - 9}{288\pi^2 \mu^2} . \tag{11.38}$$

11.3 Regularisation and Renormalisation

Above, we have encountered a divergent integral $I(-p^2)$ and in order to evaluate it anyway, we somewhat arbitrarily introduced a momentum cutoff Λ_{cut}. Gladly the cutoff has only a mild impact on the function $I(-p^2)$

$$\frac{d}{d\Lambda_{\text{cut}}} I(-p^2) = \frac{1}{16\pi^2 \Lambda_{\text{cut}}} \,. \tag{11.39}$$

In particular, the dependence on Λ_{cut} does not mix at all with the dependence on the momenta and masses! This allowed us to extract some information from $I(-p^2)$ without caring too much about the cutoff.

Regularisation Schemes.   In order to extract precise information from a QFT model at higher orders, one has to introduce a consistent regularisation scheme. Such a scheme should make all relevant quantities finite.

There are several schemes, e.g.:

- *Cutoff.* Our choice was to cut off momentum integrals at very large momenta (UV). Similarly one could cut off very small momenta (IR). Unfortunately, a cutoff is not easy to formulate consistently for all quantities. It is often used to quickly derive individual leading order results.
- *Pauli–Villars.* Replace Feynman propagators by a difference of two propagators

$$\frac{1}{p^2 + m^2 - i\epsilon} \to \frac{1}{p^2 + m^2 - i\epsilon} - \frac{1}{p^2 + M^2 - i\epsilon} \,. \tag{11.40}$$

  For large M^2 and small p^2 the second propagator is suppressed. For large p^2, however, the two propagators almost cancel. Therefore this scheme suppresses UV divergences. It is similar to a UV cutoff, but it can be applied universally to Feynman diagrams.
- *Point Splitting.* In position space, the problem of UV divergences is related to putting several fields at the same point in spacetime. By separating the field insertion points in the action by a tiny amount, UV divergences can be avoided.
- *Lattice.* For the lattice regulator one approximates infinite spacetime by a finite lattice. For finitely many degrees of freedom there cannot be divergences, neither from the UV (finite spacing), nor from the IR (finite extent).
- *Dimensional Regularisation.* The types and degrees of divergences depend crucially on the number of spacetime dimensions D. In the dimensional regularisation scheme, one works in a spacetime of dimension D, where D is taken to be an unconstrained real number. Observables become functions of D, and divergences appear as poles in the D-dependence, e.g. $1/(D-4)$. Although the definition of this scheme is somewhat abstract, it is one of the favourite ones because it works well in almost all circumstances.
- *Finite Observables.* Sometimes divergences can be avoided by considering physical observables only. In our example, one could try to argue that all observables can be deduced from $I'(s)$ which is perfectly finite. The constant term of $I(s)$ is an integration constant of $\int ds\, I'(s)$. The divergence happens to be located precisely in this undetermined coefficient.

In a regularised QFT, all observables are perfectly finite. However, they are not quite what we are interested in, we are interested in observables of the original QFT.

Renormalisation. The next step called renormalisation is to somehow absorb the divergences consistently. To that end the most important insights are the following:

- All physically relevant information and all observables for a QFT model are encoded into its quantum correlation functions.
  Example. The spectrum of asymptotic particles is encoded into time-ordered two-point functions. Moreover, the scattering matrix can be derived from the poles of higher-point functions.
- The Lagrangian and the action are devices to derive suitable correlation functions. They are not fundamental objects, in particular their parameters such as masses and coupling constants are not directly observable.[6]
  Example. The mass terms (μ) in the Lagrangian do not exactly reflect the masses (m) of asymptotic particles.
- Correlation functions depend on the so-called bare parameters of the Lagrangian. One should tune the parameter values such that the correlation functions behave as expected. At the end of the day, the numerical values of the parameters are not important.
  Example. One would adjust μ and κ such that the physical mass m has the desired physical value.
- In terms of differential geometry: The parameters of a QFT form a manifold. A Lagrangian description (with a particular regularisation, renormalisation and, where applicable, a particular gauge fixing scheme) is a chart of the manifold. The parameter values correspond to coordinates on this particular chart. There is, however, no universal meaning to coordinates without reference to the specific chart.

In this picture, renormalisation is the step to adjust the Lagrangian parameters to the physical parameters. In the regularised and finite QFT this step is well-defined. We express the bare Lagrangian parameters in terms of the physical parameters, e.g.

$$\mu = \mu(m, \Lambda_{\text{cut}}) = m + \tfrac{1}{2} m \kappa^2 I(m^2) + \dots. \tag{11.41}$$

with

$$I(m^2) = \frac{2 - \pi/\sqrt{3}}{32\pi^2} - \frac{1}{32\pi^2} \log \frac{m^2}{\Lambda_{\text{cut}}^2}. \tag{11.42}$$

Running Coupling. We can now remove the regulator by sending the regularisation parameters to some appropriately chosen limit. We shall keep the physical parameters fixed in the limit, but the resulting bare parameters may well

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[6]Some traces of the complete Lagrangian or action may remain valid in the QFT, such as the exact equations of motion and normalisation of the interacting field, as well as Noether's theorem.

be divergent

$$\mu(m, \Lambda_{\text{cut}}) \to \infty \quad \text{as} \quad \Lambda_{\text{cut}} \to \infty. \tag{11.43}$$

This by itself is not a problem, since we attribute no meaning to μ. We just need to keep in mind that the definition of bare parameters such as μ depends on a scale such as Λ_{cut}. Changing the scale must be compensated by a change in the bare parameter. This effect is called running of a coupling constant. In our case the running is governed by the equation

$$\frac{d\mu}{d\Lambda_{\text{cut}}} = \frac{m\kappa^2}{16\pi^2 \Lambda_{\text{cut}}} + \dots. \tag{11.44}$$

Such an equation is often written in logarithmic form

$$\frac{d\log\mu}{d\log\Lambda_{\text{cut}}} = \frac{\kappa^2}{16\pi^2} + \dots \tag{11.45}$$

which suggests the scaling behaviour

$$\mu \sim m(\Lambda_{\text{cut}}/m)^{\kappa^2/16\pi^2 + \cdots}. \tag{11.46}$$

Note that the dependence of the bare mass μ on the physical mass m has become non-linear by quantum effects. In the quantum field theory the mass term has acquired a so-called anomalous dimension.

A similar effect can be observed for coupling constants governing the interactions of several particles. In quantum field theory one may find an anomalous dependence on the particle momenta, e.g. $\lambda \sim (p/\Lambda)^{*\kappa^2 + \cdots}$. It means that one will measure a different effective coupling strength depending on which length or energy scale the interaction is probed (e.g. the energy of the probe photon). Naturally, one would like to define a universally valid coupling strength to appear in the Lagrangian, e.g. by considering the limit of very high or very low energies. However, in this limit, the coupling strength often diverges. Therefore one needs to define the coupling constant at a particular energy scale Λ, and the value of this coupling constant depends on Λ. This effect is called a running coupling constant.

Renormalisability. The question is whether all physical quantities remain finite in the limit $\Lambda_{\text{cut}} \to \infty$, and whether they are independent of the chosen regularisation scheme. We can only adjust one bare parameter per physical parameter, are there sufficiently many bare parameters to absorb all the divergences?

- In the case of our model the answer is yes.
- QFT models where all divergences can be absorbed are called renormalisable.
- In principle, one can introduce further terms and couplings in the Lagrangian to compensate for more and more divergences. As long as only finitely many terms are needed to absorb the divergences at all perturbative orders, the model is called renormalisable.

- Some models, such as General Relativity, appear to require infinitely many coupling constants to absorb all divergences. These models are called non-renormalisable. Here one would need infinitely many measurements to adjust infinitely many parameters, and effectively the model loses its predictive power.

Before discussing which models are renormalisable, let us consider some technical aspects of absorbing divergences into the coupling constants.

Localised Divergences. In principle, we know how to absorb divergences by writing the bare parameters of the Lagrangian as functions of the physical parameters. Let us discuss the origin of the divergence in detail.

To that end we investigate the integral $I(-p^2)$ more closely. The overall dependence on p^2 is some inverse trigonometric function.[7] However, the divergent or cutoff-dependent contribution to $I(-p^2)$ is much simpler

$$\frac{d}{d\Lambda_{\text{cut}}} I(-p^2) = \frac{1}{16\pi^2 \Lambda_{\text{cut}}} . \tag{11.47}$$

It is actually independent of p^2!

This behaviour is in fact general; divergences typically couple to polynomials of the momenta only. This statement becomes more meaningful when translated to position space: Any polynomial of the momenta translates to a localised distribution such as $\delta^4(x)$ under a Fourier transformation.

In terms of the loop integral in position space we can localise the origin of the divergence. The loop integrand diverges as r^{-4} when the two vertices are nearby at a distance of r, otherwise it is perfectly finite.

$$\tag{11.48}$$

This divergence cannot be compensated by the measure $d^4x \sim r^3 dr$, and the region $r \approx 0$ contributes an infinite amount to the integral. The divergence is therefore localised in spacetime, and it can be absorbed by a suitable local term in the Lagrangian.

$$\tag{11.49}$$

Counterterms. Indeed we see that we can absorb the divergence by a suitable definition of the bare mass term $\mu = \mu(m, \Lambda_{\text{cut}})$ in the Lagrangian. The prescription may be somewhat confusing because the integral $I(-p^2)$ depends on the mass μ, so the definition of μ apparently is implicit. For our calculations we use the interaction picture where we decided to split the Lagrangian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}$ into a free and an interaction contribution

$$\mathcal{L}_0 = -\tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}\mu^2\phi^2, \qquad \mathcal{L}_{\text{int}} = -\tfrac{1}{6}\kappa\mu\phi^3 - \tfrac{1}{24}\kappa^2\lambda\phi^4. \tag{11.50}$$

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[7]We were extremely lucky to find a simple function. Typically one finds much more complicated special functions such as polylogarithms or hypergeometric functions. And this only if one is lucky. Often the encountered integrals lead to functions which do not even have a name.

However, we are not forced to do this naive split, we are free to choose any free field as a reference. In this sense, it makes perfect sense to choose the asymptotic field with the physical mass m as a reference

$$\mathcal{L}_{\text{as}} = -\tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}m^2\phi^2. \tag{11.51}$$

This will automatically position all the poles due to Feynman propagators at the desired physical location. Now we have to add a compensating mass term to the interaction terms[8]

$$\mathcal{L}_{\text{int}} = -\tfrac{1}{2}\mu'm^2\phi^2 - \tfrac{1}{6}\kappa m\phi^3 - \tfrac{1}{24}\lambda\kappa^2\phi^4. \tag{11.52}$$

The new term is called a counterterm. Its role is to compensate all potential mass shifts due to loop effects. The mass of the reference field will thus conveniently be the physical mass to all orders.[9] In this picture, the loop integral $I(-p^2)$ is defined directly in terms of the physical mass m instead of μ.

When including the counterterm in our example, we obtain the following one-loop contributions[10]
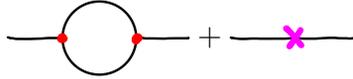
 $+$ $\tag{11.53}$

and the following corrected two-point function

$$M_2(p) = \frac{1}{p^2 + m^2 + \mu'm^2 - m^2\kappa^2 I(-p^2) - i\epsilon} + \dots. \tag{11.54}$$

We impose the consistency equation that the physical mass equals the asymptotical mass

$$m^2 = m^2 + \mu'm^2 - m^2\kappa^2 I(m^2) + \dots, \tag{11.55}$$

which is solved by $\mu' = \kappa^2 I(m^2)$. Concerning the counterterm, the rule is that whenever a bubble with two legs appears in a Feynman graph, there is a compensating counterterm.[11]
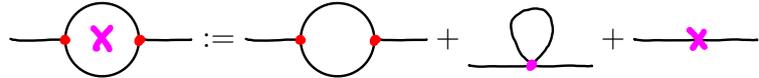
 $:=$ $+$ $+$ $\tag{11.56}$

It therefore makes sense to introduce a subtracted bubble integral

$$I_{\text{sub}}(-p^2) := I(-p^2) - I(m^2). \tag{11.57}$$

---

[8]The definition of μ', κ and λ has changed, but this does not matter since the values of the bare parameters are not directly measurable.

[9]Only the mass is stabilised by the counterterm. The two Feynman diagrams do not cancel exactly because the loop integral has a complicated dependence on p^2 whereas the counterterm is a constant function.

[10]The loop order does not necessarily refer to the literal number of loops in a Feynman graph. It makes sense to also count counterterms to loop divergences as loops. The loop order commonly refers to the order in a small coupling constant, in our case κ^2.

[11]We have reintroduced the contribution from the tadpole graph which is equivalent to a mass term with infinite mass and can therefore be absorbed entirely into a suitable redefinition of μ'.

Obviously this function is finite and satisfies

$$I_{\text{sub}}(m^2) = 0. \tag{11.58}$$

The corrected two-point function then takes the simplified form

$$M_2(p) = \frac{1}{p^2 + m^2 - m^2 \kappa^2 I_{\text{sub}}(-p^2) - i\epsilon} + \dots \tag{11.59}$$

We can even go one step further and decide to add a counterterm for the kinetic term

$$\mathcal{L}_{\text{int}} = -\tfrac{1}{2}\zeta(\partial\phi)^2 - \tfrac{1}{2}\mu' m^2 \phi^2 - \tfrac{1}{6}\kappa m \phi^3 - \tfrac{1}{24}\lambda \kappa^2 \phi^4. \tag{11.60}$$

This term changes the overall normalisation of the field $\phi(x)$, and allows us to normalise the residue of the asymptotic particle pole in the two-point function to 1 at all orders. In this case, we can drop the field strength renormalisation factor Z by setting it to 1.

In conclusion, the asymptotic Hamiltonian describes the canonically normalised fields with appropriate physical masses. The interaction Hamiltonian contains all types of allowable interactions terms. Their parameters are tuned to stabilise the masses and normalisations and to match with physical interaction processes.

Power Counting. Can we understand under which circumstances a QFT model is renormalisable? We can use a crude argument in terms of the mass dimensions of interaction terms.

A Feynman diagram evaluates to an integral over a rational function of the momenta and masses of the particles.

- Earlier, we have discussed that UV divergences of the integral are polynomials of the momenta.
- Furthermore, we can argue that particle masses in the denominators can be safely ignored for the purposes of UV divergences.
- Evidently, the coupling constants will appear as overall prefactors.

Altogether this implies that the structure of UV divergences of a loop integral is given by polynomials in the momenta and the masses

$$\frac{dI}{d\log\Lambda} \in \text{Poly}(\alpha_k, p_k, m_k). \tag{11.61}$$

The polynomials in the momenta p_k determine the appropriate local counterterm.

The class of potential divergences is restricted by the following consideration:

- The mass dimension of the integral is determined by the Feynman diagram. It is a fixed number usually bounded from above by the number of spacetime dimensions D, in our case $D = 4$.
- Suppose that all terms of the Lagrangian have mass dimension bounded from above by D. Equivalently, all coupling constants α_k have a non-negative mass dimension.

- Then the overall polynomial divergence terms must have non-negative mass dimension. The remaining mass dimension must be carried by the momenta and masses.
- This implies that there is only a very restricted set of momentum polynomials which can carry divergences. In other words, only few counterterms are needed to compensate the divergences.
- These counterterms have mass dimension bounded from above by D. The counterterms couplings in turn have non-negative mass dimension.
- Usually, there are finitely many such terms, and therefore such models are renormalisable.

Note that there is a crucial difference between coupling constants with positive mass dimension and dimensionless coupling constants. As the dimension of coupling constants accumulates, dimensionful coupling constants contribute divergences only for a specific range of low perturbative orders. When there are also dimensionless coupling constants, counterterms are required for arbitrary loop orders.

In our example, we have included all interaction terms of mass dimension bounded from above by 4. For each divergence which can possibly arise, there is a corresponding counterterm.

When considering Feynman diagrams which contain a divergent loop along with some other structures, it can be shown that the loop can be replaced by a universal counterterm corresponding to the loop to cancel the divergence. A subtle issue at higher loops are overlapping loops, where it may not be evident which counterterms to use. Gladly, it can be shown that this situation does not leave behind divergences which cannot be accounted for.

11.4 Vertex Renormalisation

Let us briefly discuss how to renormalise the remaining divergences at the one-loop level in our example.

Divergent Interactions. By power counting arguments we can derive that divergences can only appear for two-sided loops.[12] Loops with three or more legs are perfectly finite. There are only three potentially divergent terms in our model.

$$(11.62)$$

One of them we have already discussed, the other two integrals can be made finite by adding appropriate counterterms of the form ϕ^3 and ϕ^4 to the Lagrangian

$$\mathcal{L}_{\text{int}} = -\tfrac{1}{6}\kappa m \phi^3 - \tfrac{1}{24}\lambda \kappa^2 \phi^4 + \mathcal{L}_{\text{ct}},$$
$$\mathcal{L}_{\text{ct}} = -\tfrac{1}{2}\mu' m^2 \phi^2 - \tfrac{1}{6}\kappa' m \phi^3 - \tfrac{1}{24}\lambda' \kappa^2 \phi^4$$

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[12]One-sided tadpoles can be removed entirely by suitable counterterms.

$$= \text{———✗———} + \text{———✗} + \text{✗} . \tag{11.63}$$

Suitable counterterm coefficients to make all observables finite at one loop read

$$\mu' = \kappa^2 I(m^2), \qquad \kappa' = 3\kappa^3 \lambda I(m^2), \qquad \lambda' = 3\kappa^2 \lambda^2 I(m^2). \tag{11.64}$$

As discussed above, μ' is determined by a stable physical mass at m. There is no similar universal condition for the coefficients κ' and λ'; any finite shift w.r.t. the above values is permissible, it merely leads to a reparametrisation of our model.

We have already seen that the ϕ^2 counterterm effectively replaces the loop integral by a finite subtracted loop integral

$$I(-p^2) \to I_{\text{sub}}(-p^2). \tag{11.65}$$

Exactly the same replacement is achieved by our above choice of counterterms for ϕ^3 and ϕ^4.

$$\text{———●✗●———} \qquad \text{———●✗} \qquad \text{✗} \tag{11.66}$$

Three-Point Function. Let us briefly consider the resulting one-loop contributions to the three-point function.

$$F_3 = F_3^{(1)} + F_3^{(3)} + \dots,$$

$$F_3^{(1)} = \text{———●} \; ,$$

$$F_3^{(3)} = \text{———●(○)●} + \text{———●(○)✗} + \text{———✗}$$

$$+ \text{———(○)●} + \text{———✗●} $$

$$= \text{———●(○)●} + \text{———●(✗)●} + \text{———●(✗)●} . \tag{11.67}$$

The latter two terms involve bubbles only for which we know the integral already. The counterterms make both integrals finite.[13] The first triangle integral

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[13]Note that because $I_{\text{sub}}(m^2) = 0$ the internal Feynman propagator which connects the bubble to the 3-vertex is precisely cancelled, and only one Feynman propagator with associated asymptotic particle pole remains for each leg.

$\sim \int d^4\ell/\ell^6$ is UV finite by itself. In fact it is the most complicated contribution

$$
F_3^{(3)} = -i\kappa^3 m^3 (2\pi)^4 \delta^4(p_1 + p_2 + p_3)
$$
$$
\cdot \int \frac{-id^4\ell}{(2\pi)^4} \frac{1}{\ell^2 + m^2 - i\epsilon}
$$
$$
\cdot \frac{1}{(\ell + p_1)^2 + m^2 - i\epsilon} \frac{1}{(\ell - p_3)^2 + m^2 - i\epsilon} \qquad (11.68)
$$

and the only one which allows all three momenta to interact non-trivially.

Four-Point Function. There are many diagrams contributing to a four-point process. Here we can merely plot all the graphs (up to permutations of the external legs).

$$
F_4 = F_4^{(2)} + F_4^{(4)} + \dots ,
$$

$$
(11.69)
$$