Quantum Information Theory
Series 4

HS 12
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Exercise 4.1 Bloch sphere *(6 points)*

We keep going over some basics of quantum mechanics. In this exercise we will see how we may represent qubit states as points in a three-dimensional ball.
A qubit is a two level system, whose Hilbert space is equivalent to $\mathbb{C}^2$. The Pauli matrices together with the identity form a basis for 2×2 Hermitian matrices,

$$\mathcal{B} = \left\{ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \tag{1}$$

where the matrices were represented in basis $\{|0\rangle, |1\rangle\}$. Pauli matrices respect the commutation relations

$$[\sigma_i, \sigma_j] := \sigma_i\sigma_j - \sigma_j\sigma_i = 2\varepsilon_{ijk}\sigma_k, \tag{2}$$

$$\{\sigma_i, \sigma_j\} := \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}\mathbb{1}. \tag{3}$$

We will see that density operators can always be expressed as

$$\rho = \frac{1}{2}(\mathbb{1} + \vec{r} \cdot \vec{\sigma}) \tag{4}$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and $\vec{r} = (r_x, r_y, r_z), |\vec{r}| \leq 1$ is the so-called Bloch vector, that gives us the position of a point in a unit ball. The surface of that ball is usually known as the Bloch sphere.

a) Using Eq. 4 :

1) Find and draw in the ball the Bloch vectors of a fully mixed state and the pure states that form three bases, $\{|0\rangle, |1\rangle\}$, $\{|+\rangle, |-\rangle\}$ and $\{|\circlearrowleft\rangle, |\circlearrowright\rangle\}$. Use $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ and $|\circlearrowleft / \circlearrowright\rangle = (|0\rangle \pm i|1\rangle)/\sqrt{2}$.
2) Find and diagonalise the states represented by Bloch vectors $\vec{r}_1 = (\frac{1}{2}, 0, 0)$ and $\vec{r}_2 = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$.

b) Show that the operator ρ defined in Eq. 4 is a valid density operator for any vector $\vec{r}$ with $|\vec{r}| \leq 1$ by proving it fulfils the following properties:

1) Hermiticity: $\rho = \rho^\dagger$.
2) Positivity: $\rho \geq 0$.
3) Normalisation: $\mathrm{Tr}(\rho) = 1$.

c) Now do the converse: show that any two-level density operator may be written as Eq. 4.

d) Check that the surface of the ball is formed by all the pure states.

e) Discuss the analog of the Bloch sphere in higher dimensions. What can be said? For instance, where are the pure states?

Exercise 4.2 Partial trace

The partial trace is an important concept in the quantum mechanical treatment of multi-partite systems, and it is the natural generalisation of the concept of marginal distributions in classical probability theory. Given a density matrix ρ_{AB} on the bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_A = \mathrm{Tr}_B(\rho_{AB})$,

a) Show that ρ_A is a valid density operator by proving it is:

1) Hermitian: $\rho_A = \rho_A^\dagger$.
2) Positive: $\rho_A \geq 0$.
3) Normalised: $\mathrm{Tr}(\rho_A) = 1$.

b) Calculate the reduced density matrix of system A in the Bell state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right), \quad \text{where} \quad |ab\rangle = |a\rangle_{\mathrm{A}} \otimes |b\rangle_{\mathrm{B}}. \tag{5}$$

c) Consider a classical probability distribution P_{XY} with marginals P_X and P_Y.

1) Calculate the marginal distribution P_X for

$$P_{XY}(x,y) = \begin{cases} 0.5 & \text{for } (x,y) = (0,0), \\ 0.5 & \text{for } (x,y) = (1,1), \\ 0 & \text{else}, \end{cases} \tag{6}$$

   with alphabets $\mathcal{X}, \mathcal{Y} = \{0,1\}$.

2) How can we represent P_{XY} in form of a quantum state?
3) Calculate the partial trace of P_{XY} in its quantum representation.

d) Can you think of an experiment to distinguish the bipartite states of parts *b*) and *c*)?