Quantum Information Theory
Series 8

Exercise 8.1 Representations of CPTP maps

Normally, operators that map from the Hilbert space $\mathcal{H}_A$ to $\mathcal{H}_B$ are represented as matrices, such as $C = \sum_{ij} c_{ij}|i\rangle_B\langle j|_A$. We can instead represent them as vectors:

$$|C\rangle\rangle = \sum_{ij} c_{ij}|i\rangle_B|j\rangle_A. \tag{1}$$

We use this notation to denote that the operator C is represented as a vector (and therefore we use a ket), but we use the double right angle bracket to remember that this is an operator and not a state in a Hilbert space. Formally, if $C \in \mathrm{Hom}(\mathcal{H}_A, \mathcal{H}_B)$ then $|C\rangle\rangle \in \mathrm{Hom}(\mathbb{C}, \mathcal{H}_A \otimes \mathcal{H}_B)$.

a) Show that $Y \otimes X|Z\rangle\rangle = |XZY^T\rangle\rangle$, where $X \in \mathrm{Hom}(\mathcal{H}_A, \mathcal{H}_B), Y \in \mathrm{Hom}(\mathcal{H}_C, \mathcal{H}_D)$ and $Z \in \mathrm{Hom}(\mathcal{H}_C, \mathcal{H}_A)$. Note that the transpose on Y is defined in the basis chosen to represent the operators in Eq. 1.

b) Show that $\mathrm{Tr}_A(|X\rangle\rangle\langle\langle Y|) = XY^*$, where $X, Y \in \mathrm{Hom}(\mathcal{H}_A, \mathcal{H}_B)$.

c) We can use the properties (a) and (b) to now derive the Choi-Jamiołkowski representation for CPTP maps. Remember that the operator-sum representation of a map $\mathcal{E} \in \mathrm{Hom}(\mathrm{End}(\mathcal{H}_A), \mathrm{End}(\mathcal{H}_B))$ can be written as:

$$\mathcal{E}(\rho_A) = \sum_k E_k \rho_A E_k^*,$$

where $\sum_k E_k^* E_k = \mathbb{1}$. Use (a) and (b) to show that there exists a Choi-Jamiołkowski (CJ) matrix $T_{A'B} \in \mathrm{End}(\mathcal{H}_{A'B})$ such that

$$\mathcal{E}(\rho_A) = \mathrm{Tr}_A(T_{AB}(\rho_A^T \otimes \mathbb{1}_B)),$$

where A' is a copy of A, so $T_{AB} := \sum_{i,j} |i\rangle_A\langle i|_{A'} T_{A'B} |j\rangle_{A'}\langle j|_A$, and $\{|i\rangle\}_i, \{|j\rangle\}_j$ are orthonormal bases for both A and A'.

d) Show that $T_{A'B}$ from (c) can also be written as

$$T_{A'B} = (\mathbb{1}_{A'} \otimes \mathcal{E})(|\psi^+\rangle_{A'A}\langle\psi^+|),$$

where $|\psi^+\rangle_{A'A} = 1/\sqrt{d}\sum_{i=1}^d |i\rangle_{A'}|i\rangle_A = |\mathbb{1}\rangle\rangle_{A'A}$.

e) What are the CP and TP conditions on $T_{A'B}$ in the CJ picture?

f) There is another representation called the Normal representation which is defined via the following isomorphism:

$$\mathcal{N} : \mathrm{Hom}(\mathrm{End}(\mathcal{H}_A), \mathrm{End}(\mathcal{H}_B)) \mapsto \mathrm{Hom}(\mathrm{Hom}(\mathbb{C}, \mathcal{H}_{AA'}), \mathrm{Hom}(\mathbb{C}, \mathcal{H}_{BB'})),$$

where A' is a copy of A and B' is a copy of B. Show that for any CPTPM $\mathcal{E}$ there exists an operator $N_{AA' \to BB'}$ such that

$$N|\rho_A\rangle\rangle = |\mathcal{E}(\rho_A)\rangle\rangle \quad \forall \rho_A \in S_=(\mathcal{H}_A),$$

where A' is a copy of A and B' is a copy of B.

g) What is the TP condition on N?

Exercise 8.2 Measurements as unitary evolutions

Consider a measurement on a system $\mathcal{H}_A$, whose output is in $\mathcal{H}_B$ that is described by the observable $M = \sum_{x \in \mathcal{X}} xP_x$, where $\{P_x\}_x$ are projectors. If we consider the output of this measurement on a larger space, by adding an auxiliary system R, we can represent this measurement as an isometry, U, acting on A and having an output on $\mathcal{H}_B \otimes \mathcal{H}_R$, followed by a partial trace over R. By adding an ancilla to the input system, Q, the measurement can also be described by a unitary $\overline{U}$ that takes system AQ to BR, and then we get the same output as the measurement described by M by taking a partial trace over R.

a) Show that the measurement described by the observable M can be written as unitary followed by a partial trace over R. This task can be broken down into the following steps:

i) What is the Kraus operator representation of the CPTPM $\mathcal{E}$ that describes the measurement operator M?

ii) What is the Choi-Jamiołkowski (CJ) state if we write the projectors as $P_x = \sum_{k\in S_x} |k\rangle\langle k|$, where S_x is the set of indices k that we sum over for each x and $\{|k\rangle\}_k$ is an orthonormal basis?

iii) Find a purification of the CJ state. Label the purifying system as system R.

iv) Apply the inverse of the CJ isomorphism to the purified state in (iii), and show that it is of the form $U\rho_A U^*$, where U is an isometry. The inverse CJ isomorphism is the map that takes a state $\rho_{A'BR}$ as input, and outputs a map $\mathcal{F}$ defined as

$$\mathcal{F}(\rho_A) = |\mathcal{H}_A|\mathrm{Tr}_{A'}\left(\left(\sum_{i,j} |i\rangle_{A'}\langle j|_A \rho_A |i\rangle_A \langle j|_{A'}\right)\otimes \mathbb{1}_{BR}\cdot \rho_{A'BR}\right),$$

where $\{|i\rangle\}_i$ is an orthonormal basis for A and A' (similarly for $\{|j\rangle\}_j$), and $\rho_{A'BR}$ is the CJ state purified on the system R. Note that $\sum_{i,j} |i\rangle_{A'}\langle j|_A \rho_A |i\rangle_A\langle j|_{A'}$ is just the transpose and a relabeling of A to A'.

v) Show that $\mathrm{Tr}_R(\mathcal{F}(\rho_A))$ has the same output as the measurement description in (i).

vi) Finally, represent the measurement as a unitary, $\overline{U}$, acting on the input system AQ followed by a partial trace over R.

b) Give two explicit expressions (in different representations: i.e. Kraus operator, CJ, or Normal) for the two maps $\mathcal{E}$ defined by the following qubit POVMs:

1. $\mathcal{M}_1 = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$.
2. $\mathcal{M}_2 = \{p|0\rangle\langle 0|, p|1\rangle\langle 1|, (1-p)\mathbb{1}_2\}$. What is the physical interpretation of this POVM?

Exercise 8.3 Unambiguous state discrimination

Suppose you are given one of two states, ρ and σ, with equal probability, and want to distinguish them with a single measurement. We have seen that, unless the states are orthogonal ($\delta(\rho,\sigma)=1$), it is impossible to always distinguish them with certainty. We also saw that if you wanted to maximize the probability of guessing correctly, the best strategy was to measure the state in the eigenbasis of $\rho-\sigma$: you would be right with probability $\mathrm{Pr}_{\checkmark} = \frac{1}{2}(1+\delta(\rho,\sigma))$.

Now suppose you have a different goal: you will only make a guess when you are certain of which state you have, so as to never make a mistake. Formally, you will perform a measurement described by a POVM $\{M_\rho, M_\sigma, M_?\}$, such that: (1) if you obtain an outcome corresponding to M_ρ or M_σ, you know for sure that you have ρ or σ, respectively, and (2) if your outcome corresponds to $M_?$ you do not know with certainty which state you have, and you will not risk guessing.

a) We will consider only pure states $\rho = |\psi\rangle\langle\psi|, \sigma = |\phi\rangle\langle\phi|$. We want to have zero probability of guessing "ψ" when the state measured was ϕ (and vice-versa) . What does this tell us about the form of M_ψ, M_ϕ and $M_?$?

b) Maximize the probability of making a correct guess, i.e., to minimize the probability of obtaining $M_?$. Remember that you can expand one of the states in terms of the other and a vector orthogonal to it, for instance

$$|\psi\rangle = a|\phi\rangle + b|\phi^\perp\rangle, \quad |\psi^\perp\rangle = -b|\phi\rangle + a|\phi^\perp\rangle, \qquad a = \langle\psi|\phi\rangle, \quad |a|^2+|b|^2 = 1.$$

c) What happens if ψ and ϕ are given with probability q and $1-q$?