Quantum Information Theory
Series 9

Exercise 9.1 Entanglement and Teleportation

Imagine that Alice (A) has a pure state $|\psi\rangle_S$ of a system S in her lab and wants to send it to a distant lab belonging to Bob (B). Using a shared entangled state, she can do this without having to physically move the state over: she can "teleport" the state $|\psi\rangle$ to the system B that Bob controls.
Formally, we have three systems $\mathcal{H}_S \otimes \mathcal{H}_A \otimes \mathcal{H}_B$. In this exercise we will assume all three are qubits. The initial state is

$$|\psi\rangle_S \otimes \frac{1}{\sqrt{2}} \left(|0_A 0_B\rangle + |1_A 1_B\rangle\right), \tag{1}$$

i.e. A and B share a fully entangled Bell state, and the total state is separable with respect to S. We may write $|\psi\rangle_S = \alpha|0\rangle + \beta|1\rangle$.

a) In a first step, Alice will measure systems S and A jointly in the Bell basis,

$$\left\{ \begin{array}{ll} \frac{1}{\sqrt{2}}\left(|0_S 0_A\rangle + |1_S 1_A\rangle\right), & \frac{1}{\sqrt{2}}\left(|0_S 0_A\rangle - |1_S 1_A\rangle\right), \\ \frac{1}{\sqrt{2}}\left(|0_S 1_A\rangle + |1_S 0_A\rangle\right), & \frac{1}{\sqrt{2}}\left(|0_S 1_A\rangle - |1_S 0_A\rangle\right) \end{array} \right\}. \tag{2}$$

Alice then (classically) communicates the result of her measurement to Bob. What is the reduced state of Bob's system (B) for each of the possible outcomes?

b) We have seen that depending on Alice's outcome, Bob can apply certain operations to his system that will allow him to recover $|\psi\rangle$. Suppose that Alice does not manage to tell Bob the outcome of her measurement. Show that in this case he does not have any information about his reduced state and therefore does not know which operation to apply in order to obtain $|\psi\rangle$.

c) Show that this method of quantum teleportation also works for mixed states ρ_S.

d) This method even works if the state ρ_S is entangled with another system that Alice and Bob do not control. Consider a purification of ρ_S on a reference system R,

$$\rho_S = \mathrm{Tr}_R |\phi\rangle\langle\phi|_{SR}. \tag{3}$$

Show that if you apply the quantum teleportation protocol on $\mathcal{H}_S \otimes \mathcal{H}_A \otimes \mathcal{H}_B$, not touching the reference system, the final state on $\mathcal{H}_B \otimes \mathcal{H}_R$ is $|\phi\rangle$.

This implies that quantum teleportation does not only transmit the reduced state (as would for instance destroying and recreating it), but it preserves entanglement with a reference system — it transfers entanglement between S and R to B and R. This guarantees composability of teleportation: for instance, if two parts of an entangled state are transmitted via teleportation one after another, the joint system gets teleported accurately!

Exercise 9.2 Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

Read the original 1935 paper by Einstein, Podolski and Rosen titled as the question. Rewrite their argument in terms of the usual Bell setting with two qubits instead of continuous Hilbert spaces, and explain how you would prove their argument wrong.

Exercise 9.3 Majorisation and entanglement catalysts

a) Let ρ and τ be two single-qubit states characterized by their Bloch vectors,

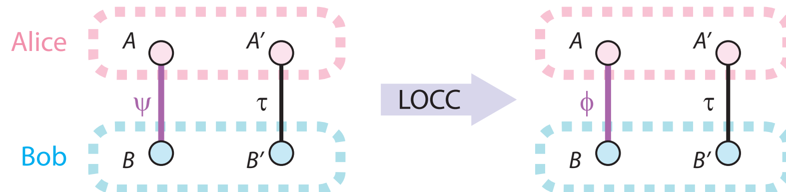
$$\rho = \frac{1}{2}(\mathbb{1} + \vec{r} \cdot \vec{\sigma}), \qquad \tau = \frac{1}{2}(\mathbb{1} + \vec{t} \cdot \vec{\sigma}).$$

Show that $\mathrm{EV}(\rho) \prec \mathrm{EV}(\tau)$ if and only if $|r| \leq |t|$. Here, $\mathrm{EV}(\rho)$ denotes the spectrum of ρ, i.e., the set of eigenvalues of ρ.

b) We perform a projective measurement described by the POVM $\{P_i\}_i$ on a state ρ. Denote the post-measurement state (not conditioned on the outcome) by ρ'. Show that $\mathrm{EV}(\rho') \prec \mathrm{EV}(\rho)$.

c) We saw that two parties that share an initial bipartite pure state $|\psi\rangle_{AB}$ can transform it into another state $|\phi\rangle_{AB}$ via LOCC if and only if $\S(\psi) \prec \S(\phi)$ (Theorem 5.3.4). Here, $\S(\psi) = \mathrm{EV}(\mathrm{Tr}_A|\psi\rangle\langle\psi|)$ is the vector formed by the eigenvalues of the reduced density matrices on each side.

There are situations where, even if that condition is not satisfied, having access to an extra entangled state $|\tau\rangle_{A'B'}$ (a *catalyst*) allows the players to transform $|\psi\rangle$ into $|\phi\rangle$ and return the catalyst untouched in the end.



Consider the following three bipartite states (on a four-level system on each side),

$$|\psi\rangle_{AB} = \sqrt{0.4}\ |00\rangle + \sqrt{0.4}\ |11\rangle + \sqrt{0.1}\ |22\rangle + \sqrt{0.1}\ |33\rangle, \qquad |\phi\rangle = \sqrt{0.5}\ |00\rangle + \sqrt{0.25}\ |11\rangle + \sqrt{0.25}\ |22\rangle,$$
$$|\tau\rangle_{A'B'} = \sqrt{0.6}\ |00\rangle + \sqrt{0.4}\ |11\rangle.$$

Check that $\S(\psi) \prec \S(\phi)$ does **not** hold, but $\S(\psi \otimes \tau) \prec \S(\phi \otimes \tau)$ does.

(Note that to compute $\S(\psi \otimes \tau)$ you have to trace out A and A' to obtain the reduced density matrix on Bob's side (BB').)