Quantum Information Theory
Solutions 09

Exercise 9.1 Quantum teleportation

Imagine that Alice (A) has state S in her lab, in pure state $|\psi\rangle_S$. She wants to send the state to Bob, who lives on the moon, without the expensive costs of shipping a coherent qubit on a space rocket. We will see that if Alice and Bob share some initial entanglement, Alice can "teleport" the state $|\psi\rangle$ to Bob's lab, using only local operations and classical communication.

Formally, we have three systems $S \otimes A \otimes B$. Alice controls systems S and A, and Bob controls B. In this exercise we will assume all three systems are qubits. The initial state is

$$|\psi\rangle_S \otimes \frac{1}{\sqrt{2}} \left(|0_A 0_B\rangle + |1_A 1_B\rangle\right), \tag{1}$$

i.e. A and B are fully entangled in a Bell state. We may write $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

(a) *In a first step, Alice will measure systems S and A jointly in the Bell basis,*

$$\left\{ \begin{array}{ll} |\phi_0\rangle = \frac{1}{\sqrt{2}} \left(|0_S 0_A\rangle + |1_S 1_A\rangle\right), & |\phi_1\rangle = \frac{1}{\sqrt{2}} \left(|0_S 0_A\rangle - |1_S 1_A\rangle\right), \\ |\phi_2\rangle = \frac{1}{\sqrt{2}} \left(|0_S 1_A\rangle + |1_S 0_A\rangle\right), & |\phi_3\rangle = \frac{1}{\sqrt{2}} \left(|0_S 1_A\rangle - |1_S 0_A\rangle\right) \end{array} \right\}. \tag{2}$$

Alice then communicates the result of her measurement to Bob: this takes two bits of classical information. What is the reduced state of Bob's system (B) for each of the possible outcomes?

Let's do this properly. First we write down the POVM elements of Alice's measurement,

$$\begin{aligned}
P_0 = |\phi_0\rangle\langle\phi_0| &= \frac{1}{2} \left(|0\rangle_S|0\rangle_A + |1\rangle_S|1\rangle_A\right)\left(\langle 0|_S\langle 0|_A + \langle 1|_S\langle 1|_A\right), \\
P_1 = |\phi_1\rangle\langle\phi_1| &= \frac{1}{2} \left(|0\rangle_S|0\rangle_A - |1\rangle_S|1\rangle_A\right)\left(\langle 0|_S\langle 0|_A - \langle 1|_S\langle 1|_A\right), \\
P_2 = |\phi_2\rangle\langle\phi_2| &= \frac{1}{2} \left(|0\rangle_S|1\rangle_A + |1\rangle_S|0\rangle_A\right)\left(\langle 0|_S\langle 1|_A + \langle 1|_S\langle 0|_A\right), \\
P_3 = |\phi_3\rangle\langle\phi_3| &= \frac{1}{2} \left(|0\rangle_S|1\rangle_A - |1\rangle_S|0\rangle_A\right)\left(\langle 0|_S\langle 1|_A - \langle 1|_S\langle 0|_A\right).
\end{aligned}$$

Note that $P_0 + P_1 + P_2 + P_3 = \mathbb{1}_{SA}$, as should be for a POVM.

Now, Alice's measurement device must have a classical registry X (like a screen or a hard drive) that saves the outcome of the measurement. A measurement corresponds to a unitary evolution on all the qubits and the classical registry,

$$|\text{ready to measure}\rangle\langle\text{ready to measure}|_X \otimes \rho_{SAB} \mapsto \sum_{k=0}^{3} |\text{outcome: } k\rangle\langle\text{outcome: } k|_X \otimes (P_k \otimes \mathbb{1}_B)\ \rho_{SAB}\ (P_k \otimes \mathbb{1}_B).$$

Notice that Bob has no access to the registry X, which tells Alice the outcome of the measurement, so from his perspective, the global state of the three qubits SAB is simply

$$\begin{aligned}
\sigma_{SAB} &= \mathrm{Tr}_X \left(\sum_{k=0}^{3} |\text{outcome: } k\rangle\langle\text{outcome: } k|_X \otimes (P_k \otimes \mathbb{1}_B)\ \rho_{SAB}\ (P_k \otimes \mathbb{1}_B) \right) \\
&= \sum_{k=0}^{3} (P_k \otimes \mathbb{1}_B)\ \rho_{SAB}\ (P_k \otimes \mathbb{1}_B)
\end{aligned} \tag{3}$$

We will look at this state again in part c). Alice, on the other hand, has access to X, and can simply read the outcome of the measurement. If she reads "outcome: k", she knows that the global state is

$$\sigma_{SAB|X=k} = \frac{(P_k \otimes \mathbb{1}_B)\ \rho_{SAB}\ (P_k \otimes \mathbb{1}_B)}{\|(P_k \otimes \mathbb{1}_B)\ \rho_{SAB}\ (P_k \otimes \mathbb{1}_B)\|}.$$

The denominator is just the probability of obtaining outcome k.

If attributing two states to the same physical system seems confusing, think of the following analogy: Bob tells Alice to buy him a lottery ticket. She does, but does not tell him the number. If you ask Bob about his (financial state) after the results come out, he will say "I am very likely not a millionaire". Alice, on the other hand, has access to the lottery result and the ticket number, so she knows exactly whether he is a millionaire or not. After she shares the ticket number with Bob, he also has this knowledge.

Now let us apply this framework to our particular initial state, $|\psi\rangle_S \otimes \frac{1}{\sqrt{2}}(|0_A 0_B\rangle + |1_A 1_B\rangle)$. After Alice obtains the outcome k, the global state becomes, from her perspective,

$$\begin{aligned}
|\gamma_k\rangle_{SAB} &= \frac{1}{\|\dots\|}\ (P_k \otimes \mathbb{1}_B)\ |\psi\rangle_S \otimes \frac{1}{\sqrt{2}}(|0_A 0_B\rangle + |1_A 1_B\rangle) \\
&= \frac{1}{\|\dots\|}\ (|\phi_k\rangle\langle\phi_k|_{SA} \otimes \mathbb{1}_B)\ |\psi\rangle_S \otimes \frac{1}{\sqrt{2}}(|0_A 0_B\rangle + |1_A 1_B\rangle)\,,
\end{aligned}$$

because the initial global state is pure. We will compute the final global state explicitly for the first outcome, $k = 0$. Remember that we can expand $|\psi\rangle_S$ in the computational basis, $|\psi\rangle_S = \alpha|0\rangle_S + \beta|1\rangle_S$. This gives us

$$\begin{aligned}
|\gamma_0\rangle_{SAB} &= \frac{1}{\|\dots\|}(|\phi_0\rangle\langle\phi_0|_{SA} \otimes \mathbb{1}_B)\ (\alpha|0\rangle_S + \beta|1\rangle_S) \otimes \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B) \\
&= \frac{1}{\|\dots\|}\left[\frac{1}{2}(|0\rangle_S|0\rangle_A + |1\rangle_S|1\rangle_A)(\langle 0|_S\langle 0|_A + \langle 1|_S\langle 1|_A) \otimes (|0\rangle\langle 0|_B + |1\rangle\langle 1|_B)\right] \\
&\qquad (\alpha|0\rangle_S + \beta|1\rangle_S) \otimes \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B) \\
&= \frac{1}{\|\dots\|}\frac{1}{2}(|000\rangle\langle 000| + |001\rangle\langle 001| + |000\rangle\langle 110| + |001\rangle\langle 111| + |110\rangle\langle 000| + |111\rangle\langle 001| + |110\rangle\langle 110| + |111\rangle\langle 111|) \\
&\qquad \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle) \\
&= \frac{1}{\frac{1}{2\sqrt{2}}\sqrt{|\alpha|^2 + |\alpha|^2 + |\beta|^2 + |\beta|^2}}\frac{1}{2\sqrt{2}}(\alpha|000\rangle + \alpha|110\rangle + \beta|001\rangle + \beta|111\rangle) \\
&= \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|110\rangle + \beta|001\rangle + \beta|111\rangle) \\
&= \frac{1}{\sqrt{2}}(|00\rangle_{SA} + |11\rangle_{SA}) \otimes (\alpha|0\rangle_B + \beta|1\rangle_B) \\
&= |\phi_0\rangle_{SA} \otimes |\psi\rangle_B.
\end{aligned}$$

The reduced state on Bob's qubit is simply $|\psi\rangle$. For anti-pedagogical purposes, we can show it explicitly by hand,

$$\begin{aligned}
\sigma_{B|X=0} &= \mathrm{Tr}_{SA}(|\gamma_0\rangle\langle\gamma_0|_{SAB}) \\
&= \frac{1}{2}\mathrm{Tr}_{SA}[(\alpha|000\rangle + \alpha|110\rangle + \beta|001\rangle + \beta|111\rangle)(\alpha^*\langle 000| + \alpha^*\langle 110| + \beta^*\langle 001| + \beta^*\langle 111|)] \\
&= \frac{1}{2}\left[|\alpha|^2\,|0\rangle\langle 0| + \alpha\beta^*|0\rangle\langle 1| + |\alpha|^2\,|0\rangle\langle 0| + \alpha\beta^*|0\rangle\langle 1| + \beta\alpha^*|1\rangle\langle 0| + |\beta|^2\,|1\rangle\langle 1| + \beta\alpha^*|1\rangle\langle 0| + |\beta|^2\,|1\rangle\langle 1|\right] \\
&= |\alpha|^2\,|0\rangle\langle 0| + \alpha\beta^*|0\rangle\langle 1| + \alpha^*\beta|1\rangle\langle 0| + |\beta|^2 \\
&= (\alpha|0\rangle + \beta|1\rangle)(\alpha^*\langle 0| + \beta^*\langle 1|) \\
&=: |b_0\rangle\langle b_0| = |\psi\rangle\langle\psi|.
\end{aligned}$$

Similarly, for the other outcomes, we obtain

$$\begin{aligned}
|\gamma_1\rangle_{SAB} &= |\phi_1\rangle_{SA} \otimes |b_1\rangle_B, & |b_1\rangle &= \alpha|0\rangle - \beta|1\rangle, \\
|\gamma_2\rangle_{SAB} &= |\phi_2\rangle_{SA} \otimes |b_2\rangle_B, & |b_2\rangle &= \beta|0\rangle + \alpha|1\rangle, \\
|\gamma_3\rangle_{SAB} &= |\phi_3\rangle_{SA} \otimes |b_3\rangle_B, & |b_3\rangle &= \beta|0\rangle - \alpha|1\rangle.
\end{aligned}$$

(b) *Suppose that Alice does not manage to tell Bob the outcome of her measurement. Show that in this case he does not have any information about the reduced state of his qubit and therefore does not know which operation to apply in order to obtain* $|\psi\rangle$.

If Bob does not know the outcome of the measurement, his knowledge about the state of the three qubits can be obtained by tracing out the registry X from the global state. This is done in Eq. 3. Bob's knowledge of qubit B is obtained by tracing out S and A,

$$\begin{aligned}
\sigma_B &= \mathrm{Tr}_{SA}\left(\sum_{k=0}^{3}(P_k \otimes \mathbb{1}_B)\ \rho_{SAB}\ (P_k \otimes \mathbb{1}_B)\right)\\
&= \sum_{q=0}^{3}(\langle\phi_q|_{SA} \otimes \mathbb{1}_B)\ \left(\sum_{k=0}^{3}(|\phi_k\rangle\langle\phi_k| \otimes \mathbb{1}_B)\ \rho_{SAB}\ (|\phi_k\rangle\langle\phi_k| \otimes \mathbb{1}_B)\right)(|\phi_q\rangle_{SA} \otimes \mathbb{1}_B)\\
&= \sum_{k=0}^{3}(\langle\phi_k|_{SA} \otimes \mathbb{1}_B)\ \rho_{SAB}\ (|\phi_k\rangle_{SA} \otimes \mathbb{1}_B)\\
&= \mathrm{Tr}_{SA}(\rho_{SAB})\\
&= \mathrm{Tr}_{SA}\left(|\psi\rangle\langle\psi|_S \otimes \frac{|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B}{\sqrt{2}}\,\frac{\langle 0|_A\langle 0|_B + \langle 1|_A\langle 1|_B}{\sqrt{2}}\right)\\
&= \frac{1}{2}(|0\rangle\langle 0|_B + |1\rangle\langle 1|_B) = \frac{\mathbb{1}_B}{2}.
\end{aligned}$$

This is a fully mixed state, which contains no information about which operation should be performed to recover $|\psi\rangle$.

(c) *Show that this method of quantum teleportation also works for mixed states ρ_S.*

First we compute the probability of Alice obatining outcome k if she measures the joint state of S and A in the Bell basis. If

$$\rho_S = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}$$

in the computational basis, then, in the Bell basis,

$$\rho_{SA} = \frac{1}{4}\begin{pmatrix} 1 & -1+2a & b+c & -b+c \\ -1+2a & 1 & b-c & -b-c \\ b+c & -b+c & 1 & -1+2a \\ b-c & -b-c & -1+2a & 1 \end{pmatrix},$$

so again the probability of obtaing each of the different outcomes is $1/4$.

Now we will see what the final state of the global system is. We can expand ρ_S in its eigenbasis, $\rho_S = \sum_i c_i|s_i\rangle\langle s_i|$ (in this case, because S is a qubit, $i = 1, 2$). The state of the global system after the measurement is

$$\begin{aligned}
\rho^k_{SAB} &= \frac{1}{\mathrm{Pr}_k}\left[(|sa^k\rangle\langle sa^k| \otimes \mathbb{1}_B)\rho^0_{SAB}(|sa^k\rangle\langle sa^k| \otimes \mathbb{1}_B)\right]\\
&= \frac{1}{4}\left[(|sa^k\rangle\langle sa^k| \otimes \mathbb{1}_B)\left(\left[\sum_i c_i|s_i\rangle\langle s_i|\right] \otimes |ab^1\rangle\langle ab^1|\right)(|sa^k\rangle\langle sa^k| \otimes \mathbb{1}_B)\right]\\
&= \sum_i c_i\left(\frac{1}{4}\left[(|sa^k\rangle\langle sa^k| \otimes \mathbb{1}_B)\left(|s_i\rangle\langle s_i| \otimes |ab^1\rangle\langle ab^1|\right)(|sa^k\rangle\langle sa^k| \otimes \mathbb{1}_B)\right]\right).
\end{aligned}$$

What is written in blue is the pure state in which the system would end up if the initial state of system S was $|s_i\rangle$, $|\Omega^k_{s_i}\rangle_{SAB}$. The reduced state on Bob's qubit is

$$\begin{aligned}
\rho^k_B &= \mathrm{Tr}_{SA}(\rho_{SAB^k})\\
&= \mathrm{Tr}_{SA}\left(\sum_i c_i|\Omega^k_{s_i}\rangle\langle\Omega^k_{s_i}|_{SAB}\right)\\
&= \sum_i c_i\ \mathrm{Tr}_{SA}\left(|\Omega^k_{s_i}\rangle\langle\Omega^k_{s_i}|_{SAB}\right)\\
&= \sum_i c_i\ |b^k_{s_i}\rangle\langle b^k_{s_i}|.
\end{aligned}$$

If Bob applies the unitary operation O_k on his qubit, he will recover ρ_S,

$$\begin{aligned} O_k \rho_B^k {O_k}^\dagger &= O_k \left(\sum_i c_i |b^k\rangle\langle b^k|_{s_i} \right) {O_k}^\dagger \\ &= \sum_i c_i \ \left(O_k |b^k_{s_i}\rangle\langle b^k_{s_i}| {O_k}^\dagger \right) \\ &= \sum_i c_i |s_i\rangle\langle s_i| = \rho_S. \end{aligned}$$

On the other hand, the reduced state of S and A after the protocol is $|sa^k\rangle$, as before.

(d) *In general, the state of S is not pure: it might be correlated with some other system that Alice and Bob do not control. Consider a purification of ρ_S on a reference system R,*

$$\rho_S = Tr_R |\psi\rangle\langle\psi|_{RS}. \tag{4}$$

Show that if you apply the quantum teleportation protocol on $S \otimes A \otimes B$, without touching the reference system, the final state on $B \otimes R$ is $|\psi\rangle$.

This implies that quantum teleportation preserves entanglement — it simply transfers it from [S and R] to [B and R].

We saw that, for an initial state $|\psi\rangle_S$, the teleportation protocol acts on systems SAB as

$$|\psi\rangle_S \otimes |\phi_0\rangle_{AB} \mapsto |\phi_0\rangle_{SA} \otimes |\psi\rangle_B$$

(we could also have $|\phi_k\rangle_{SA}$ as Alice's final state, but since she knows the outcome of her measurement, she can always apply a local unitary operation U_k on $|\phi_k\rangle_{SA}$ to rotate it to $|\phi_0\rangle_{SA}$).

Now we consider the more general case, where we want to transmit a mixed state ρ_S which may be correlated with a reference system, R: $\rho_S = \mathrm{Tr}_R |\psi\rangle\langle\psi|_{RS}$. A Schmidt decomposition (QIT script, Section 4.1.5) of $|\psi\rangle$ gives

$$|\psi\rangle_{RS} = \sum_m \sqrt{\lambda_m}\ |\alpha_m\rangle_R \otimes |\beta_m\rangle_S,$$

where $\{|\alpha_m\rangle_R\}$ and $\{|\beta_m\rangle_S\}$ are the eigenstates of the reduced density matrices ρ_R and ρ_S respectively, and $\{\lambda_m\}$ the corresponding eigenvalues.

The teleportation protocol uses only linear maps (a measurement and unitary operations), and it does not act on R, so

$$|\psi\rangle_{RS} \otimes |\phi_0\rangle_{AB} = \sum_m \sqrt{\lambda_m}\ |\alpha_m\rangle_R \otimes |\beta_m\rangle_S \otimes |\phi_0\rangle_{AB} \mapsto \sum_m \sqrt{\lambda_m}\ |\alpha_m\rangle_R \otimes |\phi_0\rangle_{SA} |\beta_m\rangle_B.$$

The reduced state on RB is the original entangled state,

$$\sum_m \sqrt{\lambda_m}\ |\alpha_m\rangle_R \otimes |\beta_m\rangle_B = |\psi\rangle_{RB}.$$

Exercise 9.2 Original EPR paper

As mentioned in the tip sheet, I would argue about the point of "element of reality". In their paper, Einstein, Podolski and Rosen define an element of reality to be any quantity that can be predicted with certainty. However, they do not restrict this prediction to depend on only locally accessible parameters: in the entanglement scenario they consider, in fact the precise measurement outcomes are only predictable based on the measurement outcomes in a space-like separated region.

This kind of definition, I would argue, does not really make sense. To see this, remember that one could always construct a reference frame in which the distant measurement actually occurs later than the one we try to make predictions about — and hence in such a reference frame, the outcome of the first measurement are not predictable with certainty.

Instead, one should really adopt a definition of physical reality that respects the causal structure of our model. Then, an element of reality is only what can be predicted with certainty based on events in the causal past of the measurement. Another example to motivate this definition is that trivially, the outcomes could always be predicted taking into account future events, but this is clearly beside the point!

Defining elements of reality this way, we can see that their argument fails: measurement outcomes of non-commuting observables do then no longer correspond to simultaneous elements of reality, even for entangled states.

Exercise 9.3 Majorisation and entanglement catalysis

a) *Let ρ and τ be two single-qubit states characterized by their Bloch vectors,*

$$\rho = \frac{1}{2}(\mathbb{1} + \vec{r}\cdot\vec{\sigma}), \qquad \tau = \frac{1}{2}(\mathbb{1} + \vec{t}\cdot\vec{\sigma}).$$

Show that $EV(\rho) \prec EV(\tau)$ if and only if $|r| \leq |t|$. Here, $EV(\rho)$ denotes the spectrum of ρ, i.e., the set of eigenvalues of ρ.

The eigenvalues of the one-qubit state ρ are $\left\{\frac{1+|\vec{r}|}{2}, \frac{1-|\vec{r}|}{2}\right\}$, and similarly for τ (see Ex. 4.1). Therefore, we have

$$\mathrm{EV}(\rho) \prec \mathrm{EV}(\tau) \Rightarrow \frac{1+|\vec{r}|}{2} \leq \frac{1+|\vec{t}|}{2} \Leftrightarrow |\vec{r}| \leq |\vec{t}|\,.$$

b) *We perform a projective measurement described by the POVM $\{P_k\}_k$ on a state ρ. Denote the post-measurement state (not conditioned on the outcome) by ρ'. Show that $EV(\rho') \prec EV(\rho)$.*

For this exercise, we consider only POVMs formed by n orthogonal projectors $\{P_k\}$: we have that $P_k^\dagger = P_k^2 = P_k$ and $P_k\rho P_\ell = 0, \forall\rho$. The (non-conditioned) post-measurement state is $\rho' = \sum_{k=1}^n P_k\rho P_k$.

We want to use Corollary 6.3.3 from the script, which tells us that $\mathrm{EV}(\rho') \prec \mathrm{EV}(\rho)$ iif there exists a set of unitaries and probabilities $\{U_j\}, \{p_j\}$, such that $\rho' = \sum_j U_j\rho U_j^\dagger$. We will try to find those unitaries. Take the family of operators $U_1, \dots, U_n$ defined as

$$U_j = \sum_{k=1}^n \mathrm{Exp}\left[2\pi i\,\frac{jk}{n}\right] P_k.$$

First we prove that they are unitary,

$$\begin{aligned}
U_jU_j^\dagger &= \left(\sum_{k=1}^n \mathrm{Exp}\left[2\pi i\,\frac{jk}{n}\right] P_k\right)\left(\sum_{\ell=1}^n \mathrm{Exp}\left[-2\pi i\,\frac{j\ell}{n}\right] P_\ell^\dagger\right)\\
&= \sum_{k=1}^n \mathrm{Exp}\left[2\pi i\,\frac{j(k-\ell)}{n}\right] P_kP_\ell\\
&= \underbrace{\sum_k P_k^2}_{\mathbb{1}} + \sum_{k\neq\ell} \mathrm{Exp}\left[2\pi i\frac{j(k-\ell)}{n}\right] \underbrace{P_kP_\ell}_{0}\\
&= \mathbb{1},
\end{aligned}$$

and similarly for $U_j^\dagger U$.

Now we will see that $\rho' = \frac{1}{n}\sum_j U_j\rho U_j^\dagger$,

$$\begin{aligned}
\sum_j U_j\rho U_j^\dagger &= \sum_j \left(\sum_{k=1}^n \mathrm{Exp}\left[2\pi i\,\frac{jk}{n}\right] P_k\right)\rho\left(\sum_{\ell=1}^n \mathrm{Exp}\left[-2\pi i\,\frac{j\ell}{n}\right] P_\ell^\dagger\right)\\
&= \sum_{j,k,\ell} \mathrm{Exp}\left[2\pi i\frac{j(k-\ell)}{n}\right] P_k\rho P_\ell\\
&= \sum_j^n \underbrace{\sum_k \; P_k\rho P_k}_{\rho'} + \sum_j\sum_{k\neq\ell} \mathrm{Exp}\left[2\pi i\frac{j(k-\ell)}{n}\right] \underbrace{P_k\rho P_\ell}_{0}\\
&= n\rho'.
\end{aligned}$$

c) *Consider the following three bipartite states (on a four-level system on each side),*

$$|\psi\rangle_{AB} = \sqrt{0.4}\;|00\rangle + \sqrt{0.4}\;|11\rangle + \sqrt{0.1}\;|22\rangle + \sqrt{0.1}\;|33\rangle, \qquad |\phi\rangle = \sqrt{0.5}\;|00\rangle + \sqrt{0.25}\;|11\rangle + \sqrt{0.25}\;|22\rangle,$$

$$|\tau\rangle_{A'B'} = \sqrt{0.6}\;|00\rangle + \sqrt{0.4}\;|11\rangle.$$

Check that $\S(\psi) \prec \S(\phi)$ *does* **not** *hold, but* $\S(\psi\otimes\tau) \prec \S(\phi\otimes\tau)$ *does.*

First we compute $\S(\psi)$ and $\S(\phi)$,

$$\begin{aligned}
\mathrm{Tr}_A(|\psi\rangle\langle\psi|) &= \mathrm{Tr}_A(0.4\ |00\rangle\langle 00| + 0.4\ |11\rangle\langle 11| + 0.1\ |22\rangle\langle 22| + 0.1\ |33\rangle\langle 33| + \sqrt{0.4\cdot 0.1}\ |00\rangle\langle 33| + \text{ other cross terms})\\
&= 0.4\ |0\rangle\langle 0| + 0.4\ |1\rangle\langle 1| + 0.1\ |2\rangle\langle 2| + 0.1\ |3\rangle\langle 3|;
\end{aligned}$$

$$\mathrm{EV}\left(\mathrm{Tr}_A(|\psi\rangle\langle\psi|)\right) = \begin{pmatrix} 0.4\\ 0.4\\ 0.1\\ 0.1 \end{pmatrix};$$

$$\begin{aligned}
\mathrm{Tr}_A(|\phi\rangle\langle\phi|) &= \mathrm{Tr}_A(0.5\ |00\rangle\langle 00| + 0.25\ |11\rangle\langle 11| + 0.25\ |22\rangle\langle 22| + \sqrt{0.5\cdot 0.25}\ |00\rangle\langle 22| + \text{ other cross terms})\\
&= 0.4\ |0\rangle\langle 0| + 0.4\ |1\rangle\langle 1| + 0.1\ |2\rangle\langle 2| + 0.1\ |3\rangle\langle 3|;
\end{aligned}$$

$$\mathrm{EV}\left(\mathrm{Tr}_A(|\psi\rangle\langle\psi|)\right) = \begin{pmatrix} 0.5\\ 0.25\\ 0.25\\ 0 \end{pmatrix}.$$

Saying that $\vec{r} \prec \vec{t}$ means that $\sum_{i=1}^k r_i \le \sum_{i=1}^k t_i, \forall k$, where the two vectors' elements are arranged in decreasing order. Let us see if that is the case for the spectra of our reduced states.

$$0.4 \le 0.5; \qquad 0.4 + 0.4 \not\le 0.5 + 0.25.$$

We now define a new notation to help answer the second part. We define a vector with the sums ($R_k = \sum_{i=1}^k r_i$), and a new symbol $\overset{E}{\le}$, that means "element-wise less or equal",i.e., $\vec{R}. \le \vec{T} \Leftrightarrow R_k \le T_k, \forall k$. Therefore $\vec{r} \prec \vec{t} \Leftrightarrow \vec{R} \overset{E}{\le} \vec{T}$. In the case of those spectra, we would have

$$\begin{pmatrix} 0.4\\ {\color{red}0.8}\\ 0.9\\ 1 \end{pmatrix} \overset{E}{\not\le} \begin{pmatrix} 0.5\\ {\color{red}0.75}\\ 1\\ 1 \end{pmatrix}.$$

Obviously, $\overset{E}{\not\le}$ means not($\overset{E}{\le}$) Why are we doing this? To simplify showing the majorization condition.

Now for the composed state, we observe that

$$\begin{aligned}
\mathrm{Tr}_{A'}(|\tau\rangle\langle\tau|) &= \mathrm{Tr}_{A'}(0.6\ |00\rangle\langle 00| + 0.4\ |11\rangle\langle 11| {\color{green}+ \sqrt{0.6\cdot 0.4}\ (|00\rangle\langle 11| + |11\rangle\langle 00|)})\\
&= 0.6\ |0\rangle\langle 0| + 0.4\ |1\rangle\langle 1|;\\
\mathrm{Tr}_{AA'}(|\phi\rangle\langle\phi|_{AB} \otimes |\tau\rangle\langle\tau|_{A'B'}) &= \mathrm{Tr}_A(|\phi\rangle\langle\phi|_{AB}) \otimes \mathrm{Tr}_{A'}(|\tau\rangle\langle\tau|_{A'B'})\\
\mathrm{EV}\left(\mathrm{Tr}_{AA'}(|\phi\rangle\langle\phi|_{AB} \otimes |\tau\rangle\langle\tau|_{A'B'})\right) &= \mathrm{EV}\left(\mathrm{Tr}_A(|\phi\rangle\langle\phi|_{AB})\right) \otimes \mathrm{EV}\left(\mathrm{Tr}_{A'}(|\tau\rangle\langle\tau|)\right)\\
&= \begin{pmatrix} 0.4\\ 0.4\\ 0.1\\ 0.1 \end{pmatrix} \otimes \begin{pmatrix} 0.6\\ 0.4 \end{pmatrix} = \begin{pmatrix} 0.24\\ 0.24\\ 0.16\\ 0.16\\ 0.06\\ 0.06\\ 0.04\\ 0.04 \end{pmatrix};
\end{aligned}$$

$$\mathrm{EV}\left(\mathrm{Tr}_{AA'}(|\psi\rangle\langle\psi|_{AB} \otimes |\tau\rangle\langle\tau|_{A'B'})\right) = \begin{pmatrix} 0.5\\ 0.25\\ 0.25\\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0.6\\ 0.4 \end{pmatrix} = \begin{pmatrix} 0.3\\ 0.2\\ 0.15\\ 0.15\\ 0.1\\ 0.1\\ 0\\ 0 \end{pmatrix}.$$

Let's check the majorization in our wonderful new notation,

$$\begin{pmatrix} 0.24 \\ 0.48 \\ 0.64 \\ 0.80 \\ 0.86 \\ 0.92 \\ 0.96 \\ 1 \end{pmatrix} \overset{E}{\leq} \begin{pmatrix} 0.30 \\ 0.50 \\ 0.65 \\ 0.80 \\ 0.90 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Success!