Exercise 4.1 Bloch sphere

The Bloch sphere is a little instrument to help us visualise the effects of quantum operations in qubit states. Here we will see how to represent states in a three-dimensional ball. You are given the lovely formula

$$\rho = \frac{1}{2}(\mathbb{1} + \vec{r} \cdot \vec{\sigma}) \tag{1}$$

and in part *a*) you only have to apply it to get a feeling for the representation of states in the Bloch sphere: you will see what we mean by "rotating" a basis, and how the purity of a state relates to its position inside the ball. Just remember that the Pauli matrices and identity matrix are represented in basis $\{|\uparrow\rangle, |\downarrow\rangle\}$. For instance the matricial representations of pure states $|\uparrow\rangle\langle\uparrow|$ and $|\downarrow\rangle\langle\downarrow|$ are

$$|\uparrow\rangle\langle\uparrow| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad |\downarrow\rangle\langle\downarrow| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Part *b*) is also fairly direct. You have to check that all reasonable Bloch vectors corresponds to valid density operators. Just apply Eq. 1 and prove those properties.
Then in part *c*) you have to prove the converse, ie. that all two-level density operators can be represented as Eq. 1 proposes. For that I suggest that you expand a general density operator ρ in the basis $\mathcal{B}$ given. Remember that you can always expand an operator A in an orthonormal basis $\{e_i\}_i$ as

$$A = \sum_i (A, e_i) e_i,$$

where the inner product (A, B) is defined as $\mathrm{Tr}(A^* B)$. Do not forget that $\mathcal{B}$ is not an orthonormal basis: you have to normalise it first. You should obtain something like

$$\rho = \frac{1}{2}\left[(\rho, \mathbb{1})\mathbb{1} + \sum_{i=x,y,z} r_i \sigma_i\right].$$

Given that ρ is a density operator, what is the value of $(\rho, \mathbb{1})$?
All you need for part *d*) is to know what $\mathrm{Tr}(\rho^2)$ is like for pure states, and relate that to $\vec{r}$.

Exercise 4.2 Partial trace

Here goes a quick introduction to partial trace. For formal definitions check pages 25-26 of the script.
Consider a composed system $\mathcal{H}_A \otimes \mathcal{H}_B$. Any state of that system can be expressed as a density matrix. For instance, in the basis $\{|a_i\rangle \otimes |b_j\rangle\}_{i,j}$, we have

$$\begin{aligned}\rho_{AB} &= \sum_{i,j} \sum_{k,l} c_{ij}^{kl} (|a_i\rangle \otimes |b_j\rangle)(\langle a_k| \otimes \langle b_l|) \\ &= \sum_{i,k} \sum_{j,l} c_{ij}^{kl} |a_i\rangle\langle a_k| \otimes |b_j\rangle\langle b_l|.\end{aligned}$$

Notation: We write composed states of the form $|x\rangle \otimes |m\rangle$ as $|x\rangle|m\rangle$ or simply $|xm\rangle$.
The (usually mixed) state of one of the subsystems can be obtained by *tracing out* the other. In practice, this means we ignore the other system (for instance, Alice may not have access to Bob's system). This is done by means of a *partial trace*,

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}). \tag{2}$$

The resulting *reduced density matrix* of subsystem A is

$$\rho_A = \sum_{i,k} \sum_j c_{ij}^{kj} |a_i\rangle\langle a_k|. \tag{3}$$

If systems A and B are independent, i.e., if ρ_{AB} is a product state, $\rho_{AB} = \rho_A \otimes \rho_B$, we have $\mathrm{Tr}_B(\rho_A \otimes \rho_B) = \rho_A \, \mathrm{Tr}(\rho_B) = \rho_A$, since density matrices are normalized.
Let us look at an example. Consider a system with two qubits A and B, prepared in the global pure state

$$|\phi\rangle = \frac{1}{\sqrt{3}} \left(|00\rangle + |10\rangle + |11\rangle\right).$$

The density matrix of the composed system is given by

$$\begin{aligned} \rho_{AB} &= |\Phi_{AB}\rangle\langle\Phi_{AB}| \\ &= \frac{1}{3} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}. \end{aligned}$$

This matrix has eigenvalues $\{1, 0, 0, 0\}$; the only non-zero eigenvalue corresponds to the eigenstate $|\phi\rangle$.
To obtain the reduced density matrix of the subsystem A, one has to evaluate the sum of Eq. 3. For instance, the coefficient α_0^1 that corresponds to the mixture $|0_A\rangle\langle 1_A|$ (in blue ahead) is given by the sum of the coefficients of the terms corresponding to $|0_A 0_B\rangle\langle 1_A 0_B|$ and $|0_A 1_B\rangle\langle 1_A 1_B|$ of the original density matrix. In the basis $\{|0_A\rangle, |1_A\rangle\}$, the reduced state is represented by the matrix

$$\rho_A = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

The colours indicate the elements of the ρ_{AB} that were summed up to each of the entries of ρ_A. Since the basis of the global system was nicely ordered, each element of the new one was calculated by the trace of the 2×2 "submatrix" that is in the corner of the original matrix indicated by the position of the desired element in the new one. The resulting state is mixed, as the reduced density matrix has two non-zero eigenvalues.
In this exercise, we will prove that the partial trace of a density matrix is still a density matrix. First, show that the partial trace of ρ_{AB} has indeed the form of Eq. 3 (use the definition from the script).
Now you are ready to solve part *a*). Hermicity and normalisation should be direct. I can give you a hint for positivity. Saying that the original density operator is semi-positive definite means that $\langle\psi|\rho_{AB}|\psi\rangle \geq 0$ for any pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. Choose $|\psi_j\rangle = |\phi\rangle \otimes |b_j\rangle$ with an arbitrary state $|\phi\rangle \in \mathcal{H}_A$ and prove that $\sum_j \langle\psi_j|\rho_{AB}|\psi_j\rangle \geq 0$ implies that ρ_A is positive.
Part *b*) is direct application of the partial trace. Check that although the original state is pure you obtain a fully mixed state when you trace out one of the systems.
In part *c*) we will treat the classical counterpart of the partial trace — marginal distributions. We have a joint probability distribution $P_{XY} = \{P_{XY}(x,y)\}_{x,y}$. You know that the marginal distribution is given by $P_X = \left\{P_X(x) = \sum_y P_{XY}(x,y)\right\}_x$, and proving positivity and normalisation should not be a problem for you. Again, applying that to the given probability distribution could not be easier.
Now you have to represent the joint probability distribution as a quantum state. Given a distribution P_X we can always represent it as a quantum state in a Hilbert space with the same dimension as the alphabet of the probability distribution:

$$\rho_{P_X} = \sum_x P_X(x)|x\rangle\langle x|$$

for some basis $\{x\}_x$. In the case of a joint distribution this becomes a state in a composed space $\mathcal{H}_X \otimes \mathcal{H}_Y$,

$$\rho_{P_{XY}} = \sum_{x,y} P_{XY}(x)|x\rangle\langle x| \otimes |y\rangle\langle y|.$$