Quantum Information Theory
Tips 7

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Quick recap on maps

The maps we care about in quantum mechanics are TPCPMs (pages 40–45 of the script), trace preserving completely positive maps. We like them because they map density operators to density operators and thus can be used to describe the physical evolution of the state of a system. Formally, a map $\mathcal{E}_A : \mathcal{H}_A \mapsto \mathcal{H}_C$ is:

a) *trace preserving* if $\text{Tr}(\rho_A) = \text{Tr}(\mathcal{E}_A(\rho_A))$,

b) *positive* if $\mathcal{E}_A(\rho_A) \geq 0$ for $\rho_A \geq 0$,

c) *completely positive* if $(\mathcal{E}_A \otimes \mathcal{I}_B)(\rho_{AB}) \geq 0$ for $\rho_{AB} \geq 0$.

At first glance one may think that *b)* and *c)* are equivalent. However, if ρ_{AB} is entangled then applying a positive operator on $\mathcal{H}_A$ may result in a non-positive operator.

Exercise 7.1 Bell-type Experiment

We will see later on the semester that Bell experiments show that quantum mechanics produces phenomena that cannot be predicted using local classical probability theory — local hidden variables. For now we will not try to compare the quantum results with what is achievable classically, but simply observe their strangeness.
The setting goes as follows. There are two parties (usually called Alice and Bob) that prepare an entangled two-qubit state. Alice keeps one of the qubits and Bob the other. No matter how far apart they are, their qubits are still entangled.
Now Alice will measure her qubit in a given basis. This will cause Bob's qubit to collapse to some state so that when he measures his qubit the measurement statistics are different from if he had decided to measure his qubit before Alice measured hers.
A well known example is when the state prepared is

$$|\psi^+\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle|0\rangle + |1\rangle|1\rangle\right)$$

for a basis $\{|0\rangle, |1\rangle\}$ in A and B.
We will see the simplest case, when Alice performs a measurement in that same basis.
The probability that she gets 0 is

$$\begin{aligned} P_A(0) &= \text{Tr}([|0\rangle\langle 0| \otimes \mathbb{1}_B] \frac{1}{2} \left(|0\rangle|0\rangle + |1\rangle|1\rangle\right) \left(\langle 0|\langle 0| + \langle 1|\langle 1|\right)) \\ &= \frac{1}{2} \end{aligned}$$

and if she obtained 0 the whole state collapses to

$$\begin{aligned} \rho_{B|A=0} &= 2\text{Tr}_A([|0\rangle\langle 0| \otimes \mathbb{1}_B] \frac{1}{2} \left(|0\rangle|0\rangle + |1\rangle|1\rangle\right) \left(\langle 0|\langle 0| + \langle 1|\langle 1|\right)) \\ &= |0\rangle\langle 0|, \end{aligned}$$

so that if Bob now measures his qubit in that basis he will always obtain $|0\rangle$. Suppose now that Alice did not tell Bob what she obtained in the measurement. In this case Alice knows that Bob has state $\rho_{B|A=0}$ but from the point of view of Bob his state is

$$\begin{aligned} \rho_B &= \text{Tr}_A(|\psi^+\rangle\langle\psi^+|) \\ &= \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|). \end{aligned}$$

If Alice and Bob were to bet on the outcome of a measurement on Bob's qubit (in basis $\{|0\rangle, |1\rangle\}$), Alice would bet on 0 and win with 100% probability, while Bob will only win with probability $\frac{1}{2}$, as from his point of view both outcomes are equally likely. In this case the probability distributions on the outcomes of Bob's measurement are

$$P_{B|A=0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad P_B = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$

It may seem at first sight that something is wrong – how can the same physical qubit be represented by two different density operators and have two different probability distributions on the same physical measurement? The answer is that that qubit is correlated to another system (Alice's qubit) and what we are looking at are the states (and probability distributions) conditioned / not conditioned on an event on that system (Alice's measurement) that is itself random: Alice is equally likely to obtain 0, in which case Bob's state collapses to $|0\rangle\langle 0|$, or 1, when get $|1\rangle\langle 1|$ on Bob's side.

There is one degree of freedom for Bob: he can choose the basis in which to measure his state. In basis $\{|0\rangle, |1\rangle\}$ he will always get 0, but if he chooses basis $\{|+\rangle, |-\rangle\}$, for instance, he may get either result with equal probability. In general, as you will prove, the *more distant* Alice's and Bob's bases are the bigger the uncertainty on Bob's measurement after Alice performed hers.

In this exercise Alice measures her state in an arbitrary basis $\left\{|\alpha\rangle, |\alpha\rangle^{\perp}\right\}$. We are dealing with a two-state system and a new basis may be defined as a rotation of a known one by an angle α. In our case, $|\alpha\rangle := \cos(\frac{\alpha}{2})|0\rangle + \sin(\frac{\alpha}{2})|1\rangle$ and $|\alpha\rangle^{\perp}$ corresponds to $|\alpha + \pi\rangle$.

To obtain the reduced state on B after the measurement on A when the outcome is known, you just have to apply the rules given above. Then you have to calculate what Bob will obtain when measuring his qubit in the "original" basis $\{|0\rangle, |1\rangle\}$. I suppose you know how to do it in the case of unknown outcome on Alice's side.