Exercise 9.1 Entanglement and Teleportation

This exercise introduces a rather spectacular result of quantum information: if two parties, Alice and Bob, share an entangled state, than they can *teleport* a state from one side to the other at the cost of the entanglement between them.

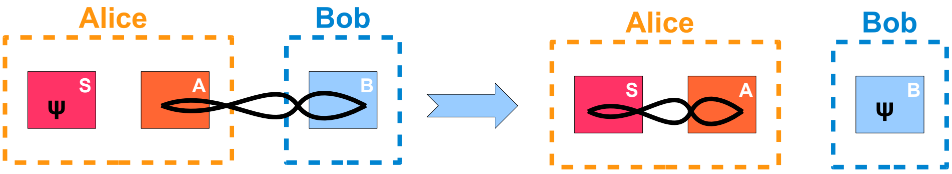

Figure 1: Quantum teleportation: in the beginning, Alice has a a qubit S in pure state $|\psi\rangle$ and a qubit A that is entangled with a qubit on Bob's side in a Bell state. By the end of the protocol, Alice's two qubits, S and A, will be entangled in a Bell state (not necessarily the same) and Bob's qubit, B, will be in state $|\psi\rangle$. The entanglement between Alice and Bob is broken when $|\psi\rangle$ is "teleported".

The setting is illustrated in Fig. 1. In her lab, Alice has a a qubit S in pure state $|\psi\rangle$ and a qubit A that is entangled with a qubit on Bob's side in a Bell state, $\frac{1}{\sqrt{2}}(|0_A 0_B\rangle + |1_A 1_B\rangle)$.
$|\psi\rangle$ is an arbitrary qubit pure state, so it may be written as $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, with $|\alpha|^2 + |\beta|^2 = 1$. The global system, $\mathcal{H}_S \otimes \mathcal{H}_A \otimes \mathcal{H}_B$, is initially in state

$$|\phi^0\rangle = (\alpha|0\rangle_S + \beta|1\rangle_S) \otimes \frac{1}{\sqrt{2}}(|0_A 0_B\rangle + |1_A 1_B\rangle)\,. \tag{1}$$

Now Alice measures her two qubits in the Bell basis,

$$\left\{|sa^k\rangle\right\}_k = \left\{ \begin{array}{ll} \frac{1}{\sqrt{2}}(|0_S 0_A\rangle + |1_S 1_A\rangle)\,, & \frac{1}{\sqrt{2}}(|0_S 0_A\rangle - |1_S 1_A\rangle)\,, \\ \frac{1}{\sqrt{2}}(|0_S 1_A\rangle + |1_S 0_A\rangle)\,, & \frac{1}{\sqrt{2}}(|0_S 1_A\rangle - |1_S 0_A\rangle) \end{array} \right\}, \tag{2}$$

obtaining outcomes 1, 2, 3 and 4 for each of the states $|sa^k\rangle$ respectively. We will see that after her measurement Bob's qubit "collapses" to a sate *very close* to $|\psi\rangle$.
The first thing you should notice is that the projectors of that measurement include the identity on B, because she is not measuring anything on that system. For instance, the projector for the first state of the Bell basis, $|sa^1\rangle$, is

$$P_1 = \frac{1}{2}\big[(|0_S 0_A\rangle + |1_S 1_A\rangle)(\langle 0_S 0_A| + \langle 1_S 1_A|)\big] \otimes \mathbb{1}_B. \tag{3}$$

Let us see what happens when Alice measures that state on her qubits, i.e. obtains outcome 1. From the foundations of quantum mechanics (page 30 of the script) you know that the final state of the global system is

$$|\phi^1\rangle = \frac{P_1|\phi^0\rangle}{\sqrt{\Pr_1}}, \tag{4}$$

where $\Pr_1$ is the probability that the outcome of her measurement is 1. You can check that for this basis all outcomes are equally likely, $\Pr_k = \frac{1}{4}, \forall_k$. We obtain

$$\begin{aligned}
|\phi^1\rangle &= \frac{1}{\sqrt{2}}\left[(|0_S 0_A\rangle + |1_S 1_A\rangle)(\langle 0_S 0_A| + \langle 1_S 1_A|) \otimes \mathbb{1}_B \right]\left[(\alpha|0\rangle_S + \beta|1\rangle_S) \otimes (|0_A 0_B\rangle + |1_A 1_B\rangle) \right] \\
&= \frac{1}{\sqrt{2}} (|0_S 0_A\rangle + |1_S 1_A\rangle) \otimes \left[\begin{array}{c} \langle \underline{0_S 0_A}| \otimes \mathbb{1}_B \left[\left(\underline{\alpha|0\rangle_S} + \beta|1\rangle_S \right) \otimes (|\underline{0_A} 0_B\rangle + |1_A 1_B\rangle) \right] \\ + \langle \underline{1_S 1_A}| \otimes \mathbb{1}_B \left[\left(\alpha|0\rangle_S + \underline{\beta|1\rangle_S} \right) \otimes (|0_A 0_B\rangle + |\underline{1_A} 1_B\rangle) \right] \end{array} \right] \\
&= \frac{1}{\sqrt{2}} (|0_S 0_A\rangle + |1_S 1_A\rangle) \otimes \left[\alpha|0\rangle_B + \beta|1\rangle_B \right] \quad = \quad |as^1\rangle \otimes |\psi\rangle_B.
\end{aligned}$$

I hope the rainbow above has not blinded you and that you managed to follow what happened there and how we ended up with a fully correlated Bell state on $S \otimes A$ that is decoupled from B, where we find $|\psi\rangle$. The key, of course, lies in the strong correlations between A and B. If you repeat this procedure to all possible outcomes(part *a*) of the exercise), you should obtain the table

Alice's outcome	Alice's state	Bob's state	Bob performs
1	$\vert as^1\rangle = \frac{1}{\sqrt{2}} (\vert 0_S 0_A\rangle + \vert 1_S 1_A\rangle)$	$\vert b^1\rangle = \alpha\vert 0\rangle + \beta\vert 1\rangle$	O_1
2	$\vert as^2\rangle = \frac{1}{\sqrt{2}} (\vert 0_S 0_A\rangle - \vert 1_S 1_A\rangle)$	$\vert b^2\rangle = \alpha\vert 0\rangle - \beta\vert 1\rangle$	$O_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
3	$\vert as^3\rangle = \frac{1}{\sqrt{2}} (\vert 0_S 1_A\rangle + \vert 1_S 0_A\rangle)$	$\vert b^3\rangle = \beta\vert 0\rangle + \alpha\vert 1\rangle$	O_3
4	$\vert as^4\rangle = \frac{1}{\sqrt{2}} (\vert 0_S 1_A\rangle - \vert 1_S 0_A\rangle)$	$\vert b^4\rangle = \beta\vert 0\rangle - \alpha\vert 1\rangle$	O_4

Not always it happens that the state of Bob's system is exactly $|\psi\rangle$. For instance, when Alice obtains outcome 2, his qubit goes to state $\alpha|0\rangle - \beta|1\rangle$, and he would have to perform a one-qubit operation on his system to recover $|\psi\rangle$. In this case, he would have to flip the sign of $|1\rangle$, applying the unitary represented by O_2 in the computational basis. But all of this you should have covered in the lectures.
Of course, Bob only knows what operations to apply because he knows the state $|b^k\rangle$ of his qubits, and he knows that because Alice told him the outcome k of her measurement. What if Alice had not told him the outcome? In that case, Bob would have to try to guess what the state of his qubit. He knows that all measurement outcomes were equally likely, and that for each of them he would have a different state. Fortunately, in quantum mechanics we have a way of describing probabilistic mixtures of pure states — with density matrices. The state Bob has after Alice's measurement is, from his point of view, $\rho = \sum_k \frac{1}{4}|b^k\rangle\langle b^k|$. In part *b*) you have to show that when Bob does not know the outcome of the measurement, he cannot have any idea of what his state is or how to recover $|\psi\rangle$, i.e. $\rho = \frac{1}{2}\mathbb{1}_B$. This tells us that the quantum teleportation protocol can only work if Alice uses a (possibly classical) communication channel to share some information with Bob (the outcome of her measurement).
Notice that when Alice and Bob teleport the state of one qubit, they lose their entanglement, and therefore cannot repeat the protocol to teleport anything else. Impressive as it is, quantum teleportation comes with a cost. So far we have only seen how to teleport a pure state. One may wonder what happens if the state Alice tries to teleport is entangled with a reference system R that she does not control. Would the final state on Bob's side be entangled with R in the same way? The answer is, swimmingly, yes (Fig. 2).
In parts *c*) and *d*) of the exercise you are asked to prove that more formally. You can start by considering that every mixed state can be expanded in its eigenbasis, $\rho_S = \sum_i p_i |i\rangle\langle i|_S$, with $|i\rangle = \alpha_i|0\rangle + \beta_i|1\rangle$. Check that the protocol works for such a state. You can, for instance, show what happens when Alice measures her two qubits in the Bell basis and obtains outcome 2. Remember that the final state of the whole system is given by

$$\frac{1}{\Pr_2}\left(|as^2\rangle\langle as^2| \otimes \mathbb{1}_B \right) \left[\rho_S \otimes \frac{1}{2} (|0_A 0_B\rangle + |1_A 1_B\rangle)(\langle 0_A 0_B| + \langle 1_A 1_B|) \right] \left(|as^2\rangle\langle as^2| \otimes \mathbb{1}_B \right). \tag{5}$$

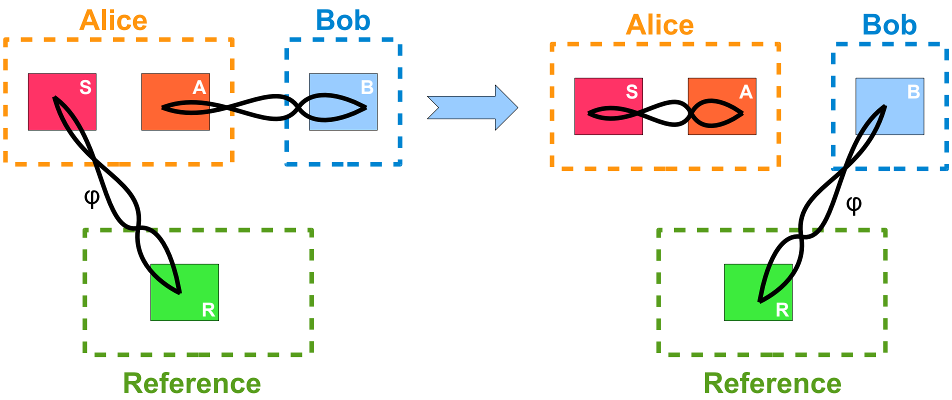

Figure 2: Quantum teleportation preserves entanglement. If Alice teleports a mixed state ρ_S that is entangled with a reference system R, $\rho_S = \text{Tr}_R |\phi\rangle\langle\phi|_{SR}$, not only the final state on Bob's side will be ρ but it will be entangled with R in the exact same way as before, $\rho_B = \text{Tr}_R |\phi\rangle\langle\phi|_{BR}$.

Verify that in order to recover ρ on his qubit, Bob only needs to apply the unitary O_2 as before. Argue that it also works for the remaining measurement outcomes. This implies, in particular, that the protocol is linear: you did not have to use the convexity of density operators ($\sum_i p_i = 1$) to prove this result. Now we are ready to tackle correlations between ρ_S and an external system R. Try making a Schmidt decomposition of the pure state $|\phi\rangle$ of $\mathcal{H}_S \otimes \mathcal{H}_R$. You should get something like $|\phi\rangle_{SR} = \sum_i p_i |i\rangle_S \otimes |i\rangle_R$. If you call the quantum teleportation protocol $\mathcal{E}$, apply $\mathcal{E} \otimes \mathcal{I}_R$ on that state and use the linearity of $\mathcal{E}$ you should obtain the result we are looking for.

Exercise 9.2 Completeness of Quantum Theory - Original Bell paper

Keep the following questions in mind (they are only ideas though, take them with a grain of salt - maybe this way of thinking helps you, maybe not):

1. In the paper they consider continuous Hilbert spaces, and look at position and momentum observables. Could you translate their argument into the usual 2-party, 2-qubit space?

2. Think about the definition of physical reality. In particular, who needs to be able to predict with certainty the relevant outcomes? Try to introduce light cones and causal structure into the argument by Einstein, Podolsky and Rosen and reformulate their assumptions and definitions using this language.

Exercise 9.3 Majorization and entanglement catalysts

To learn more about majorization, check Section 6.3 of the script, and this book by Nielsen and Vidal: `http://www.rintonpress.com/journals/qic-1-1/vidal.pdf` .
Quick recap: say that ρ and σ are d-dimensional states with eigenvalues $\{a_i\}_i$ and $\{b_i\}_i$, respectively. Then $\text{EV}(\rho) \prec \text{EV}(\tau)$ means that

$$\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i, \forall k \leq d.$$

In part $a)$ you just have to apply this to the qubit case, $d = 2$, and see the consequences for the Bloch vectors of the two states. Express the eigenvalues of ρ (and σ) as a function of $|\vec{r}|$ (and $|\vec{t}|$) and the result should be direct.
For part $b)$ 3. we apply a (von Neumann) projective measurement on ρ. The post-measurement state, not conditioned on the outcome, is just $\rho' = \sum_k P_k \rho P_k$, with $\sum_k P_k^\dagger P_k = \mathbb{1}$. We consider only orthonormal projectors, so $P_k^\dagger = P_k^2 = P_k$. Note also that $P_k P_\ell = P_\ell P_k = \delta_{k\ell} P_k$. An example of such a POVM is just

to measure in an orthonormal basis, or a coarse-graining of that measurement, like $P_1 = \sum_{x=1}^{5} |x\rangle\langle x|, P_2 = \sum_{x=6}^{8} |x\rangle\langle x|$, for an o.n. basis $\{|x\rangle\}_{x=1}^{8}$.
We want to show that $\mathrm{EV}(\rho') \prec \mathrm{EV}(\rho)$. Here is a suggestion on how to prove it. Say there are n projectors $\{P_k\}_k$ in total. Create a family of operators $U_1, U_2, \dots, U_n$ defined as

$$U_j = \sum_{k=1}^{n} \mathrm{Exp}\left[2\pi i \, \frac{jk}{n}\right] P_k,$$

and check that they are unitaries. Now see that

$$\sum_j \sum_k U_j \rho U_j^\dagger = n\rho'.$$

Finally, use Corollary 6.3.3 from the script to prove the desired result.
For an alternative proof, start by proving the statement for two orthogonal projectors, and then use induction to obtain the general case.
Part c) is pretty straight-forward, and pretty strange! **Extra:** Can you think of an explicit procedure that Alice and Bob may use to take $|\psi\rangle|\tau\rangle \to |\phi\rangle|\tau\rangle$ via LOCC?