Exercise 1. ***Getting to Know the Qubit***

As seen in the lecture, a qubit is an abstract notion implemented by a quantum mechanical two-level system. It can be in any state $\rho \in \mathcal{S}\left(\mathbb{C}^2\right)$, where $\mathcal{S}(\mathscr{H})$ are the positive operators of unit trace on $\mathscr{H}$, also called density operators.

The state ρ can be represented by its *Bloch sphere representation*, a vector $\vec{a}$ inside the unit ball in $\mathbb{R}^3$. The correspondance is given by

$$\rho = \frac{1}{2}\left(\mathbb{1} + \vec{a}\cdot\vec{\sigma}\right) \ ; \tag{1a}$$

$$a_i = \mathrm{tr}\left(\rho\,\sigma_i\right) \ , \tag{1b}$$

where $\{\sigma_i\}$ are the Pauli matrices.

The canonical basis vectors of $\mathbb{C}^2$ are given by $|0\rangle = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $|1\rangle = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$.

(a) Give the Bloch vectors corresponding to the following pure states, and draw them on the Bloch sphere.

$$|0\rangle \quad ; \quad |1\rangle \quad ; \quad |\pm\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right) \quad ; \quad |\pm i\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \pm i|1\rangle\right) \ .$$

(b) Give the Bloch vectors corresponding to the following states, and draw them on the Bloch sphere:

$$\frac{1}{2}\mathbb{1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \quad ; \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad ; \quad \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \ .$$

Rotations on the Bloch sphere correspond to unitaries on $\mathbb{C}^2$ (up to an irrelevant global phase). Recall that the unitaries correspond to a change of orthonormal bases. A unitary U satisfies $U^\dagger U = UU^\dagger = \mathbb{1}$.

(c) Pure states on the qubit (obviously) have the form $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. Where are the pure states located in the Bloch sphere representation?

Hint. Use a unitary...

(d) [1] Prove relations (1a) and (1b), i.e. that to each vector in the Bloch sphere corresponds a quantum state and vice versa. Argue in particular that the length of the Bloch vector $\vec{a}$ satisfies $|\vec{a}| \leqslant 1$.

Hint. The Pauli matrices satisfy $\sigma_i\sigma_j = \delta_{ij}\mathbb{1} + i\varepsilon_{ijk}\sigma_k$, *as well as* $\mathrm{tr}\left(\sigma_i\sigma_j\right) = 2\delta_{ij}$.

[1] Extra question with more calculations...

Exercise 2. ***Measurements on the Qubit***

A measurement of the qubit along the Z axis will give the result either $+1$ or -1, yielding $+1$ with probability $\frac{1}{2} + \frac{1}{2}a_z$.

(a) Which quantum mechanical observable corresponds to this measurement?

(b) What happens if you measure the qubit, in state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, along the Z axis? What is the probability to measure $+1$? and -1? In each case, what is the post-measurement state?

Consider now a quantum mechanical observable A, written in diagonal form as $A = a|a\rangle\langle a| + a'|a'\rangle\langle a'|$, with eigenvalues a and a' and with two orthonormal eigenvectors $|a\rangle$ and $|a'\rangle$.

(c) What happens in the Bloch sphere representation if you measure a qubit with an observable A instead of measuring it along the Z axis?

Hint. A unitary might help again...

(d) Let's prepare now the qubit in state $|+\rangle$. What are the outcome probabilities for a measurement along the Z axis?

(e) Now let's prepare randomly the qubit in either state $|0\rangle$ or $|1\rangle$ with probability $1/2$. Write the density operator for this system. What are the outcome probabilities for a measurement along the Z axis?

(f) Consider again the two systems given in (d) and (e), but now measure them along the X axis. What happens?