Exercise 1. ***Three Qubit Bit Flip Code.***

Let $|\psi\rangle = \alpha|000\rangle + \beta|111\rangle$, with $|\alpha|^2 + |\beta|^2 = 1$, be an encoding of the qubit $\alpha|0\rangle + \beta|1\rangle$.

(a) Compute the eigenvalues and eigenvectors of the observables $Z_1Z_2 := Z \otimes Z \otimes \mathbb{I}$ and $Z_2Z_3 := \mathbb{I} \otimes Z \otimes Z$.

(b) Perform the measurement of the observable Z_1Z_2 followed by the observable Z_2Z_3 on the faulty state $X_1|\psi\rangle$ with $X_1 := X \otimes \mathbb{I} \otimes \mathbb{I}$. What are the corresponding outcomes, measurements probabilities and post-measurement states?

(c) Do the same calculations for the states $|\psi\rangle$, $X_2|\psi\rangle$ and $X_3|\psi\rangle$.

(d) How can a single bit-flip error in $|\psi\rangle$ be corrected by using the information obtained by the measurements of Z_1Z_2 and Z_2Z_3?

Exercise 2. ***Shor code.***

Let $|\psi\rangle$ be the nine qubit Shor-encoding of the qubit $\alpha|0\rangle + \beta|1\rangle$. Assume that $|\psi\rangle$ is exposed to a noise process which introduces a bit and a phase flip error on the fourth qubit yielding the faulty state $Z_4X_4|\psi\rangle$.

(a) Perform the measurement Z_4Z_5 followed by Z_5Z_6 on $Z_4X_4|\psi\rangle$. What are the corresponding outcomes, measurement probabilities, and post-measurement states? Infer from the measurement results where the bit flip operation has to be applied in order to correct one of the errors.

(b) Measure the observables $X_1X_2X_3X_4X_5X_6$ and $X_4X_5X_6X_7X_8X_9$ on the bit-flip corrected state of part (a). What are the corresponding outcomes, measurements probabilities and post-measurement states? What can be inferred about the error(s) left in the state from the measurement results?

(c) Apply the operator $Z_4Z_5Z_6$ to the resulting state of the previous part. What is the final state?

(d) How would you correct the error $Z_iX_i|\psi\rangle$, where the position i of the error is not known?

Exercise 3. ***Quantum Fourier Transform.***

The *quantum Fourier transform* is just a discrete Fourier transform written in terms of kets. Given an orthonormal basis $\{|0\rangle \dots |N-1\rangle\}$, it is defined to be the linear operator with the following action on the basis states,

$$|j\rangle \longmapsto \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \mathrm{e}^{2\pi i\, jk/N} |k\rangle \ . \tag{1}$$

(a) Argue that this operation is unitary.

(b) Compute the Fourier transform of the n-qubit state $|0 \dots 0\rangle$.