Exercise 1. *Three Qubit Bit Flip Code.*

Let $|\psi\rangle = \alpha|000\rangle + \beta|111\rangle$, with $|\alpha|^2 + |\beta|^2 = 1$, be an encoding of the qubit $\alpha|0\rangle + \beta|1\rangle$.

(a) Compute the eigenvalues and eigenvectors of the observables $Z_1Z_2 := Z \otimes Z \otimes \mathbb{I}$ and $Z_2Z_3 := \mathbb{I} \otimes Z \otimes Z$.

(b) Perform the measurement of the observable Z_1Z_2 followed by the observable Z_2Z_3 on the faulty state $X_1|\psi\rangle$ with $X_1 := X \otimes \mathbb{I} \otimes \mathbb{I}$. What are the corresponding outcomes, measurements probabilities and post-measurement states?

(c) Do the same calculations for the states $|\psi\rangle$, $X_2|\psi\rangle$ and $X_3|\psi\rangle$.

(d) How can a single bit-flip error in $|\psi\rangle$ be corrected by using the information obtained by the measurements of Z_1Z_2 and Z_2Z_3?

Solution.

(a) The spectral decomposition of the Pauli matrix Z is given by $Z = (+1)|0\rangle\langle 0| + (-1)|1\rangle\langle 1|$. The eigenvectors of Z_1Z_2 corresponding to the eigenvalue $+1$ are therefore $|000\rangle, |001\rangle, |110\rangle$ and $|111\rangle$. The eigenvectors of Z_1Z_2 corresponding to the eigenvalue -1 are given by $|010\rangle, |011\rangle, |100\rangle$ and $|101\rangle$.

For the observable Z_2Z_3 we obtain the eigenvectors $|000\rangle, |100\rangle, |011\rangle$ and $|111\rangle$ corresponding to the eigenvalue $+1$ and the eigenvectors $|010\rangle, |110\rangle, |001\rangle$ and $|101\rangle$ corresponding to the eigenvalue -1.

(b) Applying the bit flip on the first qubit gives the state $X_1|\psi\rangle = \alpha|100\rangle + \beta|011\rangle$. Measuring the observable Z_1Z_2 then yields -1 with probability 1 as $X_1|\psi\rangle$ is an element of the space spanned by the eigenvectors corresponding to the eigenvalue -1 (see previous item). Furthermore, this implies that the state $X_1|\psi\rangle$ is not altered by this measurement.

Measuring Z_2Z_3 yields the outcome $+1$ with probability 1 as $X_1|\psi\rangle$ is an element of the space spanned by the eigenvectors corresponding to the eigenvalue $+1$. Again, the state is not changed by this measurement.

(c) By using the same reasoning as above we can show that

- $|\psi\rangle$: measuring Z_1Z_2 yields $+1$ and Z_2Z_3 yields $+1$.
- $X_2|\psi\rangle$: measuring Z_1Z_2 yields -1 and Z_2Z_3 yields -1.
- $X_3|\psi\rangle$: measuring Z_1Z_2 yields $+1$ and Z_2Z_3 yields -1.

The states are not changed by any of these measurements.

(d) The previous two items imply that the following strategy corrects a single bit flip error:

- Measuring $+1$, $+1 \Rightarrow$ do nothing
- Measuring -1, $+1 \Rightarrow$ apply X_1
- Measuring -1, $-1 \Rightarrow$ apply X_2
- Measuring $+1$, $-1 \Rightarrow$ apply X_3

Exercise 2. *Shor code.*

Let $|\psi\rangle$ be the nine qubit Shor-encoding of the qubit $\alpha|0\rangle + \beta|1\rangle$. Assume that $|\psi\rangle$ is exposed to a noise process which introduces a bit and a phase flip error on the fourth qubit yielding the faulty state $Z_4X_4|\psi\rangle$.

(a) Perform the measurement Z_4Z_5 followed by Z_5Z_6 on $Z_4X_4|\psi\rangle$. What are the corresponding outcomes, measurement probabilities, and post-measurement states? Infer from the measurement results where the bit flip operation has to be applied in order to correct one of the errors.

(b) Measure the observables $X_1X_2X_3X_4X_5X_6$ and $X_4X_5X_6X_7X_8X_9$ on the bit-flip corrected state of part (a). What are the corresponding outcomes, measurements probabilities and post-measurement states? What can be inferred about the error(s) left in the state from the measurement results?

(c) Apply the operator $Z_4Z_5Z_6$ to the resulting state of the previous part. What is the final state?

(d) How would you correct the error $Z_iX_i|\psi\rangle$, where the position i of the error is not known?

Solution.

(a) First note that the faulty state is given by

$$\begin{aligned} Z_4X_4|\psi\rangle &= \alpha\left(\frac{|000\rangle+|111\rangle}{\sqrt{2}}\right)\otimes\left(\frac{-|100\rangle+|011\rangle}{\sqrt{2}}\right)\otimes\left(\frac{|000\rangle+|111\rangle}{\sqrt{2}}\right) \\ &+ \beta\left(\frac{|000\rangle-|111\rangle}{\sqrt{2}}\right)\otimes\left(\frac{-|100\rangle-|011\rangle}{\sqrt{2}}\right)\otimes\left(\frac{|000\rangle-|111\rangle}{\sqrt{2}}\right) . \end{aligned}$$

The projector P_+^{45} which projects onto the eigenbasis of Z_4Z_5 corresponding to the eigenvalue $+1$ is given by (see Exercise 1)

$$P_+^{45} = \mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I}\otimes(|00\rangle\langle 00|+|11\rangle\langle 11|)\otimes\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I} ,$$

where each identity operator $\mathbb{I}$ acts on a single qubit. Similarly, for the projector P_-^{45} corresponding to the eigenvalue -1 has the following expression:

$$P_-^{45} = \mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I}\otimes(|01\rangle\langle 01|+|10\rangle\langle 10|)\otimes\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I} .$$

It is not hard to see that $Z_4X_4|\psi\rangle \in \text{range}(P_-^{45})$, i.e. that $Z_4X_4|\psi\rangle$ is an element of the space spanned by the eigenvectors of Z_4Z_5 corresponding to the eigenvalue -1. As $Z_4X_4|\psi\rangle \in \text{range}(P_-^{45})$ it holds that $P_-^{45}Z_4X_4|\psi\rangle = Z_4X_4|\psi\rangle$ and therefore the state is not changed by the measurement. For the measurement Z_5Z_6 we obtain the following projectors

$$\begin{aligned} P_+^{56} &= \mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I}\otimes(|00\rangle\langle 00|+|11\rangle\langle 11|)\otimes\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I} \\ P_-^{56} &= \mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I}\otimes(|01\rangle\langle 01|+|10\rangle\langle 10|)\otimes\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I} . \end{aligned}$$

This time we have that $Z_4X_4|\psi\rangle \in \text{range}(P_+^{56})$ and therefore obtain the measurement result $+1$ with probability 1. Again, the state is not altered by the measurement.

As we have outcomes -1 and $+1$ we can conclude, by using Exercise 1, that we have to apply X_4 in order to correct the bit flip on the fourth qubit.

(b) Applying the bit flip operation X_4 on the faulty state yields the bit flip corrected state

$$X_4(Z_4X_4|\psi\rangle) = -Z_4X_4X_4|\psi\rangle = -Z_4|\psi\rangle , \tag{S.1}$$

where we used that X_4 and Z_4 anti-commute and that $X_4X_4 = \mathbb{I}$.

Let $|+\rangle := 1/\sqrt{2}(|0\rangle+|1\rangle)$ and $|-\rangle := 1/\sqrt{2}(|0\rangle-|1\rangle)$. Note that $X|+\rangle = (+1)|+\rangle$ and $X|-\rangle = (-1)|-\rangle$. The projector P_+ corresponding to the eigenbasis of the observable $X_1X_2X_3$ belonging to the eigenvalue $+1$ is then given by

$$P_+ = |+++\rangle\langle +++| + |+--\rangle\langle +--| + |-+-\rangle\langle -+-| + |--+\rangle\langle --+| .$$

And similarly, for the projector belonging to the eigenvalue -1 we obtain

$$P_- = |---\rangle\langle ---| + |++-\rangle\langle ++-| + |-++\rangle\langle -++| + |+-+\rangle\langle +-+| .$$

The corresponding projectors for the measurement $X_1X_2X_3X_4X_5X_6$ are then given by

$$\begin{aligned} P_+^{1..6} &= P_+\otimes P_+\otimes\mathbb{I}^{\otimes 3} + P_-\otimes P_-\otimes\mathbb{I}^{\otimes 3} \\ P_-^{1..6} &= P_+\otimes P_-\otimes\mathbb{I}^{\otimes 3} + P_-\otimes P_+\otimes\mathbb{I}^{\otimes 3} . \end{aligned}$$

By using that $1/\sqrt{2}(|000\rangle + |111\rangle) \in \text{range}(P_+)$ and $1/\sqrt{2}(|000\rangle - |111\rangle) \in \text{range}(P_-)$ we can conclude that with probability 1 the measurement outcome -1 is obtained, and therefore the state is not changed by the measurement.

For the measurement $X_4X_5X_6X_7X_8X_9$ we obtain the projectors

$$\begin{aligned} P_+^{4..9} &= \mathbb{I}^{\otimes 3} \otimes P_+ \otimes P_+ + \mathbb{I}^{\otimes 3} \otimes P_- \otimes P_- \\ P_-^{4..9} &= \mathbb{I}^{\otimes 3} \otimes P_+ \otimes P_- + \mathbb{I}^{\otimes 3} \otimes P_- \otimes P_+ \ , \end{aligned}$$

and therefore, we obtain the outcome -1 with probability 1. Again, the state is not changed.

As we have the measurement outcomes -1 and -1 we can conclude, by using Exercise 1 and the fact that a phase flip in the $\{|0\rangle, |1\rangle\}$ basis is a bit flip in the $\{|+\rangle, |-\rangle\}$ basis, that a phase flip error has occurred in the second block of three qubits.

(c) Note that $(Z \otimes Z \otimes \mathbb{I})|000\rangle = |000\rangle$ and $(Z \otimes Z \otimes \mathbb{I})|111\rangle = |111\rangle$. Applying $Z_4Z_5Z_6$ on the state given in (S.1) then yields

$$(Z_4Z_5Z_6)(-Z_4|\psi\rangle) = -Z_5Z_6|\psi\rangle = -|\psi\rangle \ .$$

Hence, we have recovered the initial state $|\psi\rangle$ (with a global phase).

(d) The same procedure as above can be used.

(i) Measure $Z_1Z_2, Z_2Z_3, Z_4Z_5, Z_5Z_6, Z_7Z_8, Z_8Z_9$. This leaves the state unchanged, and then given the measurement outcomes (syndrome), we can correct the bit flip error. More specifically, we have the four cases in part (b) and (c) of Exercise 1 in either block 123, 456, or 789, and can determine where to apply an X operator.

(ii) For the phase flip we can measure $X_1X_2X_3X_4X_5X_6$ and $X_4X_5X_6X_7X_8X_9$. This determines which block the Z error occurs in. Specifically, -1 +1 eigenvalues mean the Z error is in block 123, -1 -1 eigenvalues mean the Z error is in block 456, +1 -1 eigenvalues mean the Z error is in block 789. By applying a Z operation to each qubit in the block with an error $-|\psi\rangle$ is left.

Exercise 3. *Quantum Fourier Transform.*

The *quantum Fourier transform* is just a discrete Fourier transform written in terms of kets. Given an orthonormal basis $\{|0\rangle \dots |N-1\rangle\}$, it is defined to be the linear operator with the following action on the basis states,

$$|j\rangle \longmapsto \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \mathrm{e}^{2\pi i\, jk/N} |k\rangle \ . \tag{1}$$

(a) Argue that this operation is unitary.

Solution. Let us show that the inverse of this operation is also its adjoint. We know that the inverse Fourier transform is given by

$$|k\rangle \longmapsto \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \mathrm{e}^{-2\pi i\, jk/N} |j\rangle \ . \tag{S.2}$$

(This can be verified explicitely by plugging it into (1) and checking that we get back $|j\rangle$ again.)

The matrix elements of the transformation (1), written as a linear operator U, are given by $u_{kj} = \langle k|U|j\rangle = \frac{1}{\sqrt{N}}\mathrm{e}^{2\pi i\, jk/N}$. The inverse transform has the matrix elements $v_{jk} = \frac{1}{\sqrt{N}}\mathrm{e}^{-2\pi i\, jk/N} = u_{kj}^*$, which is also the adjoint of U.

(b) Compute the Fourier transform of the n-qubit state $|0\dots0\rangle$.

Solution. It suffices to set $k = 0$ in (1) to notice that the Fourier transform of $|0 \ldots 0\rangle$ is simply the completely uniform vector (i.e. a uniform superposition of all basis states),

$$U|0 \ldots 0\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} |k\rangle \ . \tag{S.3}$$