In quantum state tomography, one often makes statements based on the likelihood function $\mathcal{L}(\rho)$. *The function is defined for a given dataset obtained during the tomography procedure, and is a function on state space. The likelihood is defined as the probability that from the state ρ, the quantum measurements you performed would give you the actual dataset that you observed,*

$$\mathcal{L}(\rho) = \Pr[\text{observed data} \,|\, \rho] \ . \tag{1}$$

It is often convenient for various applications to consider the logarithm of this function, the loglikelihood $\ell(\rho) = \ln \mathcal{L}(\rho)$.

More generally, the likelihood and loglikelihood are basic quantities for parameter estimation in statistics (not only in quantum mechanics).

One simple tomography procedure, known as maximum likelihood estimation, *is to take measurements, write down the likelihood function, and report the maximum of this function as being a suitable candidate for an estimation ρ_{MLE} of what the quantum state of our system was. (Technically, one usually prefers for large amount of data to numerically maximize the loglikelihood, since the latter is concave.)*

Tips— The probability inside the likelihood function (1) is given to us by elementary quantum mechanics. It is the probability of getting the sequence of outcomes which actually occurred by performing the measurements that we did, while we vary the state ρ on which we hypothetically perform those measurements.

For example, if one performed a single measurement of a 2-outcome POVM $\{Q, \mathbb{1} - Q\}$, and obtained "Q" as outcome (=the dataset), then the likelihood function would be simply

$$\mathcal{L}(\rho) = \mathrm{tr}(Q\rho) \ . \tag{T.1}$$

Exercise 1. *State Tomography of a Coin.*

Suppose that you have a biased coin, that gives "heads" with probability p and gives "tails" with probability $1-p$. You would like to estimate this bias, based on a finite number of coin flips.

Suppose that you flip the coin N times, and obtain f_h times heads and $f_t = N - f_h$ times tails.

(a) *Write down the likelihood function $\mathcal{L}(p)$ and the loglikelihood function $\ell(p)$ for these measurements, as a function of f_h and f_t. Plot these functions in the interval $p \in [0, 1]$.*
(b) *What would you report as the "true" bias of the coin?*

Remark. This problem is identical to measuring only σ_z on an unknown qubit, and estimating the Z-coordinate of its state on the Bloch sphere.

Exercise 2. *Quantum State Tomography on a Qubit.*

Suppose you have a qubit on which you perform tomographic measurements in order to determine its state ρ_{true}. You measure in the Pauli basis, performing n_x measurements of the σ_x observable, n_y of the σ_y observable, and n_z for σ_z. You accumulate statistics by keeping track of the number f_i^+ of "+1"'s measured for the observable σ_i, and the number f_i^- of "−1"'s observed for that observable.

(a) *Write down the likelihood function and the loglikelihood function for this procedure, as a function of the f_i^s's. Show that the loglikelihood function is concave.*
(b) *The* linear inversion *state ρ_{LI} is the one that reproduces the correct probabilities as given by the observed frequences. Calculate this state, which must satisfy*

$$\mathrm{tr}(\rho_{\mathrm{LI}} \Pi_i^s) = \frac{f_i^s}{n_i} \ , \tag{2}$$

where Π_i^s is the POVM effect corresponding to measuring s on the observable σ_i (i.e., it is the projector onto the s eigenspace of σ_i). Is this state always well defined?

Hint. Go to the Bloch sphere picture.

Tips— The *linear inversion* state is the naïve state estimate: the idea is to look at the observed frequencies f_i^s, and identify them with the corresponding probabilities of obtaining s when measuring your true state with observable σ_i. That is, you're looking for a state ρ_{LI} that satisfies (2).

In the Bloch sphere with X/Y/Z measurements, this is particularly simple as one simply needs to find the state that has the right expectation values on each of these measurements (why?). Recall that the coordinates of a state in the Bloch sphere are exactly its expectation values for the corresponding observable σ_i.

(c) *Let ρ_{MLE} be the maximum likelihood estimate. Show that if the linear inversion gives a state ρ_{LI} inside the state space, it coincides with the maximum likelihood estimate ρ_{MLE}.*