Integrability in QFT and AdS/CFT

Lecture Notes

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0 Overview

0.1 Introduction

What is integrability?

- ... a peculiar feature of some theoretical physics models.
- ... makes calculations in these models much more feasible in principle and in practice; it is also known as solvability.
- ... allows to compute some quantities exactly and analytically rather than approximately and numerically.
- ... is a hidden enhancement of symmetries which constrain the motion substantially or completely.
- ... is the absence of chaotic motion.
- ... is a colourful mixture of many subjects and techniques from mathematics to physical phenomena.
- ... a lot of fun.

Which classes of models are integrable?

- some classical mechanics models, e.g.: free particle, harmonic oscillator, spinning top, planetary motion, ...[1]
- some $(1+1)$-dimensional classical field theories, e.g.: KdV, sine-Gordon, Einstein gravity, sigma models on coset spaces, classical magnets, string theory.
- some quantum mechanical models, e.g. the quantum versions of the above classical mechanics models.
- some $(1+1)$-dimensional quantum field theories, e.g. most of the quantum counterparts of the above classical field theories, except cases where integrability is spoiled by quantum effects.
- some 2-dimensional models of statistical mechanics, e.g. 6-vertex model, 8-vertex model, alternating sign matrices, loop models, Ising model, ...
- $D = 4$ self-dual Yang–Mills theory.
- $D = 4$, $N = 4$ maximally supersymmetric Yang–Mills theory in the planar limit and the AdS/CFT dual string theory on $AdS_5 \times S^5$.

One observes that integrability is a phenomenon largely restricted to two-dimensional systems. There are some higher-dimensional exceptions, but most of them have some implicit two-dimensionality (self-duality, planar limit).

[1] Most models discussed in lectures and textbooks are in fact integrable, most likely because they can be solved easily and exactly.

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0.3 Literature

- G. Arutyunov, "Students Seminar: Classical and Quantum Integrable Systems"
- O. Babelon, D. Bernard, M. Talon, "Introduction to Classical Integrable Systems", Cambridge University Press (2003)
- N. Beisert, "The Dilatation Operator of $\mathcal{N} = 4$ super Yang-Mills Theory and Integrability", Phys. Rept. 405, 1–202 (2004), `http://arxiv.org/abs/hep-th/0407277`.
- N. Reshetikhin, "Lectures on the integrability of the 6-vertex model", `http://arxiv.org/abs/1010.5031`.
- L.D. Faddeev, "How Algebraic Bethe Ansatz Works for Integrable Model", `http://arxiv.org/abs/hep-th/9605187`.
- P. Zinn-Justin, "Six-Vertex, Loop and Tiling Models: Integrability and Combinatorics"
- D. Bernard, "An Introduction to Yangian Symmetries", Int. J. Mod. Phys. B7, 3517–3530 (1993), `http://arxiv.org/abs/hep-th/9211133`.
- V. Chari, A. Pressley, "A Guide to Quantum Groups", Cambridge University Press (1995).
- N. Beisert et al., "Review of AdS/CFT Integrability: An Overview", `http://arxiv.org/abs/1012.3982`.
- ...

1 Classical Integrability

Here we discuss integrability for a system of classical mechanics with finitely many degrees of freedom. Although this will not be the main subject of this course, it is very instructive because there is a clear notion of integrability in this case which lays the foundation for the more elaborate cases of field theory and quantum mechanics discussed later.

1.1 Hamiltonian Mechanics

We start by defining a classical mechanics system in Hamiltonian formulation. It consists of a phase space $\mathcal{M}$ of dimension $2n$ and a Hamiltonian function $H : \mathcal{M} \to \mathbb{R}$. Phase space is defined by a set of coordinates q^k and momenta p_k with $k = 1, \dots, n$.

A solution of the system is a curve $(q^k(t), p_k(t))$ in phase space which obeys the Hamiltonian equations of motion

$$\dot{q}^k = \frac{\partial H}{\partial p_k}\,, \qquad \dot{p}_k = -\frac{\partial H}{\partial q^k}\,. \tag{1.1}$$

It is convenient to introduce Poisson brackets which map a pair of functions F, G on phase space to another function on phase space[1]

$$\{F, G\} := \frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q^k} - \frac{\partial F}{\partial q^k}\frac{\partial G}{\partial p_k}\,. \tag{1.2}$$

The Poisson brackets are anti-symmetric and they obey the Jacobi identity

$$\big\{\{F, G\}, H\big\} + \big\{\{G, H\}, F\big\} + \big\{\{H, F\}, G\big\} = 0. \tag{1.3}$$

The Poisson brackets allows to write the equations of motion in a compact and uniform fashion as

$$\frac{d}{dt}\, q^k = \{H, q^k\}, \qquad \frac{d}{dt}\, p_k = \{H, p_k\}. \tag{1.4}$$

More generally, the time-dependence of a function $F(q, p, t)$ evaluated on a solution $(q^k(t), p_k(t))$ reads[2]

$$\frac{d}{dt}\, F = \frac{\partial F}{\partial t} + \{H, F\}. \tag{1.5}$$

[1] The Poisson brackets are often defined by specifying the canonical relations $\{p^k, q_l\} = \delta^k_l$ along with the trivial ones $\{p^k, p^l\} = \{q_k, q_l\} = 0$.

[2] To make proper sense of the above equations of motion one should introduce the coordinate functions $q^k(q, p, t) := q^k$, $p_k(q, p, t) := p_k$ whose partial derivatives w.r.t. time vanish, $\partial q^k/\partial t = \partial p_k/\partial t = 0$.

1.2 Integrals of Motion

For a time-independent Hamiltonian, $\dot{H} = 0$, the function H is an *integral of motion* or *conserved quantity*

$$\frac{d}{dt} H = \frac{\partial H}{\partial t} + \{H, H\} = 0. \tag{1.6}$$

The immediate benefit is that solutions are constrained to a hypersurface of $\mathcal{M}$ defined by $H = E = \text{const.}$ (constant energy). It is therefore easier to find solutions.

Depending on the model, further (time-independent)[3] integrals of motion F_k can exist

$$\frac{d}{dt} F_k = \{H, F_k\} \stackrel{!}{=} 0. \tag{1.7}$$

This gives additional constraints $F_k = f_k = \text{const.}$ and motion takes place on an even lower-dimensional hypersurface which is called a *level set*

$$\mathcal{M}_f := \{x \in \mathcal{M}; F_k(x) = f_k\}. \tag{1.8}$$

By construction, the Hamiltonian H is among them and one may identify $F_1 = H$.

Additional simplifications come about when the integrals are *in involution* or *(Poisson) commute*

$$\{F_k, F_l\} = 0. \tag{1.9}$$

This allows to consistently define solutions (q, p) depending on several time variables t^k such that time-dependence for any function G is determined by[4]

$$\frac{d}{dt^k} G = \frac{\partial G}{\partial t^k} + \{F_k, G\}. \tag{1.10}$$

Finding integrals of motion is all but straight-forward:

- They are often found by trial and error based on a suitable ansatz.
- Noether's theorem implies the existence of a conserved quantity for each global symmetry of the system.[5]

1.3 Liouville Integrability

A system with $2n$-dimensional phase space $\mathcal{M}$ is called (Liouville) integrable if it has

[3] Throughout this course we will implicitly assume that functions of phase space have no explicit time dependence.

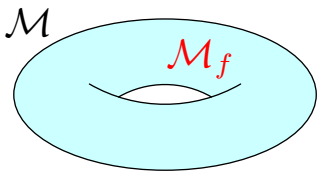
[4] Consistency requires that d/dt^k commutes with d/dt^l which is guaranteed by $\{F_k, F_l\} = 0$ by means of the Jacobi identity.

[5] Additional conserved quantities can often be regarded as a consequence of additional hidden symmetries of the system.

- n independent[6]
- everywhere differentiable
- integrals of motion F_k
- in involution, $\{F_k, F_l\} = 0$.

Such a system is solvable by quadratures, i.e. it suffices to solve a finite number of algebraic equations and integrals.

For integrable systems the following theorem holds: If the level set $\mathcal{M}_f$ is compact, it is diffeomorphic to the n-dimensional torus T^n, the so-called *Liouville torus*.[7]



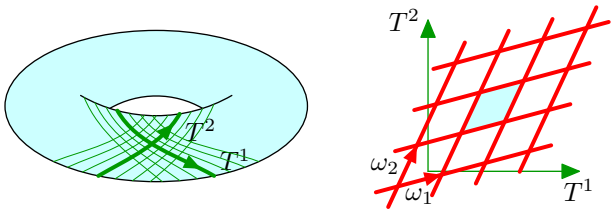
(1.11)

For an integrable system, we can define a set of n time functions T^k on phase space such that $\{F_k, T^l\} = \delta_k^l$. These differential equations define the T^k on each level set. Furthermore, the time functions can be defined across the level sets by imposing the differential equations $\{T^k, T^l\} = 0$. Suitable functions can be constructed thanks to the Jacobi identities. Altogether we have

$$\{F_k, T^l\} = \delta_k^l, \qquad \{F_k, F_l\} = \{T^k, T^l\} = 0, \tag{1.12}$$

which tells us that the map $(q^k, p_k) \to (T^k, F_k)$ is a canonical transformation.

Note that the time functions are in general multiple-valued on phase space. Going around a non-trivial cycle of a level set, the times jump by a definite amount,[8] given by the period matrix. In that sense, the time functions T^k are uniquely defined on the universal cover $\mathbb{R}^n$ of the level sets. Conversely, the level set is the quotient of $\mathbb{R}^n$ by the lattice defined by the periods.



(1.13)

A useful corollary of integrability is that motion on the level set torus is linear since

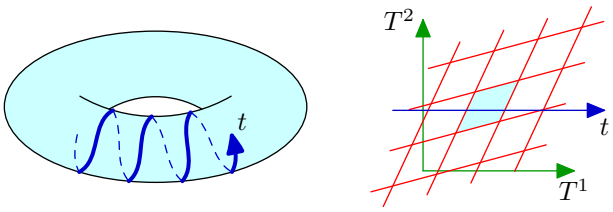
$$\{H, T^k\} = \{F_1, T^k\} = \delta_1^k, \tag{1.14}$$

[6] A function of phase space is called independent of a set of functions if it cannot be written as a function of the values of the other functions. For instance, the total angular momentum $J^2 = J_x^2 + J_y^2 + J_z^2$ is dependent on the components $\{J_x, J_y, J_z\}$ of the angular momentum. Moreover, a constant function is always dependent, even on the empty set of functions.

[7] This theorem follows by considering the vector fields associated to the integral of motions F_k defined by the operation $\{F_k, \cdot\}$. By construction, the vector fields act within the level set and commute with each other. It is well-known that a compact manifold of dimension n which admits n commuting vector fields is diffeomorphic to the torus T^n.

[8] The differential equations determine the time functions locally up to a constant shift.

where we have assumed that H is F_1. In other words, the solution in the new coordinates is $T^1 = t$ and all other T^k and all F_k are constant.



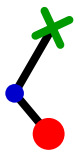
(1.15)

It would make sense to discuss some basic integrable models in detail, in particular, how to solve them exactly. However, many of them can be found in textbooks of classical mechanics, and it makes no sense to repeat the treatment here.

1.4 Comparison of Classes

Locally we can always define functions (T^k, F_k) such that $\{F_k, T^l\} = \delta_k^l$, $\{F_k, F_l\} = \{T^k, T^l\} = 0$ with the Hamiltonian as $H = F_1$. This means that any system of classical mechanics can be considered integrable in a sufficiently small patch of phase space. In that sense, integrability is a property which depends strongly on the global structure of phase space.

Chaos. However, it may be impossible to continue the integrals of motion consistently to all regions of phase space. In other words, when following the level set along a solution one may end up in the initial region with the level set misaligned with the original one. This is a hallmark feature of *chaotic motion*. An exponential divergence of solutions essentially implies that tracing out nearby solutions would lead to highly complicated hypersurfaces spread out wildly across phase space which could not possibly be described as level sets of globally differentiable functions F_k. Most dynamical systems with more than one degree of freedom, i.e. a phase space of dimension 4 or higher, are chaotic. An example of a chaotic system is the double pendulum.[9]



(1.16)

Integrability. For an integrable system the hypersurfaces match up well globally due to their definition as a level set of differentiable functions F_k. As discussed above, one finds linear motion on the level set torus. Since the torus typically has several periods which are rationally incompatible, the motion of the system is quasi-periodic. All (time-independent) dynamical systems with one degree of freedom are integrable. Further examples include the multi-dimensional HO, the spinning top, planetary motion and classical integrable spin chains.

[9] Its two constituent pendulums will alternate between oscillatory and rotational motion with a seemingly random pattern of repetitions.

Super-Integrability. Some systems have more than n integrals of motion, but evidently only n of them can be in involution. These systems are called *super-integrable*. Here some of the periods of the tori are rationally compatible and therefore the orbits partially close. For a maximally super-integrable system with $2n-1$ integrals of motion the orbits close and the motion is truly periodic. Dynamical systems with one degree of freedom are in fact maximally super-integrable and therefore have periodic motion. Further examples are Kepler's planetary motion,[10] the spinning top[11] and multi-dimensional harmonic oscillators with rationally compatible frequencies.

For non-integrable systems, there may be further useful distinctions that could be made:

- n integrals of motion which are not (all) in involution,
- less than n (but more than one) integrals of motion,
- regions of phase space of a chaotic system which admit integrability.

In this lecture series we will only be interested in the fully integrable cases. Super-integrability may occur accidentally, but we will not pay attention to it.

1.5 Structures of Integrability

In the following we shall introduce two important structures for integrable models that will (later in the context of integrable field theories) help us investigate and solve the model.

Lax Pair. Integrable systems are often formulated in terms of a *Lax pair*. A Lax pair is a pair of square matrices L, M whose entries are functions of phase space. The characteristic property of Lax pairs is that the equation

$$\frac{d}{dt}L = [M, L] \tag{1.17}$$

is equivalent to the complete set of equations of motion.

If a Lax pair exists for a classical mechanics system, the matrix L can be used to generate a tower of integrals of motion F_k

$$F_k = \operatorname{tr} L^k. \tag{1.18}$$

These quantities are trivially conserved due to the cyclicity of the trace

$$\frac{d}{dt}F_k = k \operatorname{tr} L^{k-1}[M, L] = 0. \tag{1.19}$$

[10] In addition to the angular momentum there is the Runge-Lenz vector which is orthogonal to the angular momentum vector. This amounts to 5 integrals of motion and maximal super-integrability.

[11] The spinning top has 3 integrals of motion in involution H, J^2 and J_z and is therefore integrable. Among the two further components of the angular momentum vector only one is independent of J^2 and J_z.

For systems with finitely many degrees of freedom, only finitely many of the generated charges can be independent.

Alternatively, the Lax equation is equivalent to the statement that time evolution of L is generated by a similarity transformation. Therefore the eigenvalue spectrum and the characteristic polynomial of L are conserved. Note that the latter is a function of the F_k.

Having a Lax pair formulation of integrability is very convenient, but

- inspiration is needed to find it,
- its structure is hardly transparent,
- it is not at all unique,
- the size of the matrices is not immediately related to the dimensionality of the system.

Therefore, the concept of Lax pairs does not provide a means to decide whether any given system is integrable (unless one is lucky to find a sufficiently large Lax pair).

Classical r-matrix. For integrability we not only need sufficiently many global integrals of motion F_k, but they must also be in involution, $\{F_k, F_l\} = 0$. In the formulation of integrability in terms of a Lax pair $L, M \in \mathrm{End}(V)$, this is equivalent to the statement

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2]. \tag{1.20}$$

The statement is defined on the tensor product space $\mathrm{End}(V \otimes V)$ of two matrices, and the *classical r-matrix* r_{12} is a particular element of this space whose entries are functions on phase space.[12] Furthermore, $L_1 := L \otimes 1$, $L_2 := 1 \otimes L$, and $r_{21} := P(r_{12})$ denotes the permutation of the two spaces for the r-matrix. Note that the r-matrix is by no means uniquely defined by the equation.[13] Much like for the Lax pair, there is no universal method to obtain the r-matrix.

From the above equation it follows straight-forwardly that

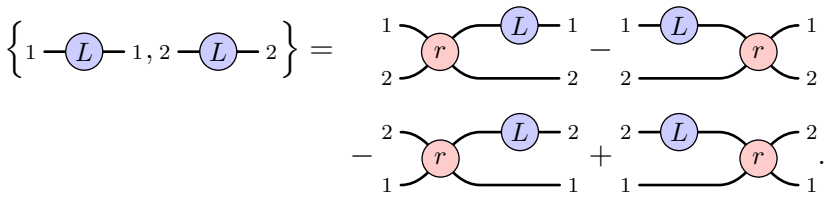
$$\{\mathrm{tr}\, L^k, \mathrm{tr}\, L^l\} = 0. \tag{1.21}$$

There is a useful graphical representation of the equation where matrices are objects with one ingoing and one outgoing leg. Connecting two legs corresponds to a product of matrices, whereas two matrices side by side correspond to a tensor product. Consequently, the classical r-matrix will be an object with two ingoing

[12] The tensor product $A \otimes B$ of two matrices with elements $A^a{}_c$, $B^b{}_d$ has the elements $(A \otimes B)^{ab}{}_{cd} = A^a{}_c B^b{}_d$, where ab and cd are combined indices enumerating a basis for the tensor product space.

[13] For example, one can add an operator of the form $1 \otimes X + L \otimes Y$ where X and Y are arbitrary matrices.

and outgoing legs, and the above equation reads



(1.22)

Many relationship can be conveniently expressed and proved using this graphical notation. We shall make extensive use of it in the context of integrable spin chains.

Example. Consider a harmonic oscillator with frequency ω. A Lax pair is given by

$$L = \begin{pmatrix} +p & \omega q \\ \omega q & -p \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & -\frac{1}{2}\omega \\ +\frac{1}{2}\omega & 0 \end{pmatrix}. \tag{1.23}$$

The Lax equation is equivalent to the equation of motion of the harmonic oscillator

$$\dot{p} = -\omega^2 q, \qquad \omega\dot{q} = \omega p. \tag{1.24}$$

The resulting integrals of motion read

$$\begin{aligned} F_1 &= 0, \\ F_2 &= 2p^2 + 2\omega^2 q^2 = 4H, \\ F_3 &= 0, \\ F_4 &= 2(p^2 + \omega^2 q^2)^2 = 8H^2, \\ &\dots . \end{aligned} \tag{1.25}$$

Here F_1 and F_3 are trivial and can be ignored. The first and only non-trivial integral of motion F_2 is the Hamiltonian. The higher even powers are merely powers of the Hamiltonian which are not independent integrals of motion.

For this system, a classical r-matrix is given by

$$r_{12} = \frac{1}{q}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \frac{1}{q}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{1.26}$$

The above commutators with the Lax matrix L then agrees precisely with the Poisson brackets

$$\{L_1, L_2\} = \omega \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \omega \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.27}$$

2 Integrable Field Theory

2.1 Classical Field Theory

Next we consider classical mechanics of a one-dimensional field $\varphi(x)$. Together with time-evolution, this amounts to a problem of $(1+1)$-dimensional fields $\varphi(t,x)$. The phase space for such models is infinite-dimensional,[1] thus integrability requires infinitely many integrals of motion in involution. Comparing infinities is subtle, so defining integrability requires care. Since there is no clear notion of integrability for field theories, we will be satisfied with the availability of efficient constructive methods for solutions. Whether or not a model is formally integrable will be of little concern.

Most random field theory models are clearly non-integrable, but there are several well-known models that are integrable:

- Korteweg–de Vries (KdV) equation

$$\dot{u} = 6uu' - u'''. \tag{2.1}$$

  This is perhaps the prototype integrable field theory. It models surface waves in shallow water.
- Non-linear Schrödinger equation

$$i\dot{\psi} = -\psi'' + 2\kappa|\psi|^2\psi. \tag{2.2}$$

- sine-Gordon equation (relativistic)

$$\ddot{\phi} - \phi'' + \frac{m^2}{\beta}\sin(\beta\phi) = 0. \tag{2.3}$$

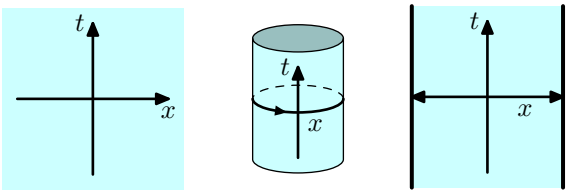
  This equation has many generalisations: non-linear sigma models on coset spaces.
- classical Heisenberg magnet (Landau–Lifshitz equation)

$$\dot{\vec{S}} = -\kappa\vec{S}\times\vec{S}'', \qquad \vec{S}^2 = 1. \tag{2.4}$$

In addition to the bulk equations of motion, a complete definition of the model also requires the specification of *boundary conditions*. The most common choices are:

- infinite spatial extent with rapidly decaying fields (or derivatives) as $x \to \infty$,
- closed or periodic boundary conditions with $x \equiv x + L$,
- open boundaries with Dirichlet or Neumann conditions $\phi = \text{const.}$ or $\phi' = 0$.

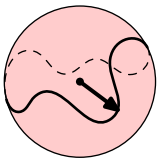
[1] Note that a time slice is a field which can be Taylor or Fourier expanded leading to infinitely many independent coefficients.



(2.5)

Boundary conditions may also be twisted in some way or combined differently.

Heisenberg Magnet. The classical Heisenberg magnet is a model of a one-dimensional magnetic material with a magnetisation or spin vector field $\vec{S}(t,x)$. The spin vector has a constant length, $\vec{S}^2 = 1$.[2]



(2.6)

The energy depends on the alignment of nearby spins, the simplest ansatz is[3]

$$H = \frac{\kappa}{2} \int dx\, \vec{S}'^2. \tag{2.7}$$

A suitable Poisson structure is[4]

$$\{S^a(x), S^b(y)\} = \varepsilon^{abc} S^c(x) \delta(x-y). \tag{2.8}$$

The equations of motion are the so-called Landau–Lifshitz equations

$$\dot{\vec{S}}(x) = \{H, \vec{S}(x)\} = -\kappa \vec{S}(x) \times \vec{S}''(x). \tag{2.9}$$

The system is formulated without making reference to a preferred vector. Therefore it has a global rotational symmetry $\vec{S}(x) \mapsto R\vec{S}(x)$ with $R \in \mathrm{SO}(3)$. This leads to a Noether current $\vec{J}_\alpha$ and associated Noether charge $\vec{Q}$

$$\vec{J}_t = \vec{S}, \qquad \vec{J}_x = -\kappa \vec{S} \times \vec{S}', \qquad \vec{Q} = \int dx\, \vec{S}. \tag{2.10}$$

The current and charge are conserved $\vec{J}'_x - \dot{\vec{J}}_t = 0$ and $\dot{\vec{Q}} = 0$ provided the field $\vec{S}$ satisfies the equations of motion.

This model is integrable, therefore there are many additional integrals of motion.

[2]The field $\vec{S}(t,x)$ can also be viewed as the evolution of a one-dimensional curve on a two-dimensional sphere S^2.

[3]Note that $\vec{S} \cdot \vec{S}' = 0$ due to $\vec{S}^2 = 1$.

[4]The above Hamiltonian along with the Poisson structure follows from a Lagrange function which is somewhat subtle.

Lax Connection. We want to set up a Lax pair to describe the integrals of motion for the field theory. In field theory we require infinitely many conserved quantities so the Lax pair has to be infinite-dimensional. Alternatively, we can set up a Lax pair with an additional continuous parameter λ, the so-called *spectral parameter*. Taylor expansion in λ then leads to an infinite tower of conserved quantities.

Furthermore, the we prefer to formulate in terms of local objects in a field theory. Therefore introduce the *Lax connection* $A_\alpha(\lambda; t, x)$

$$\begin{aligned} A_x(\lambda) &= -\frac{i}{\lambda}\,\vec{\sigma}\cdot\vec{S}, \\ A_t(\lambda) &= \frac{i\kappa}{\lambda}\,\vec{\sigma}\cdot(\vec{S}\times\vec{S}') + \frac{2i\kappa}{\lambda^2}\vec{\sigma}\cdot\vec{S}. \end{aligned} \tag{2.11}$$

In our case, the Lax connection is a 2×2 matrix-valued field whose entries depend on λ and are functions of phase space. Here $\vec{\sigma}$ is the triplet of 2×2 traceless hermitian Pauli matrices. The Lax connection satisfies the flatness condition for all λ

$$\dot{A}_x(\lambda) - A_t'(\lambda) + [A_x(\lambda), A_t(\lambda)] = 0 \tag{2.12}$$

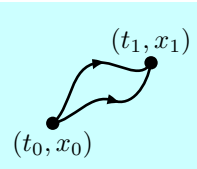
provided that the equations of motion hold and $\vec{S}^2 = 1$.

As always, the Lax connection is not unique. However, a useful recipe to construct it, is to make an ansatz in terms of the components of a Noether current, J_x and J_t in our case, and constrain the coefficients by means of the flatness condition.

It is convenient to work with the Lax connection using the language of differential forms. It is a $\mathfrak{su}(2)$ connection one-form $A(\lambda) = A_x(\lambda)dx + A_t(\lambda)dt$ which obeys $dA(\lambda) = A(\lambda)\wedge A(\lambda)$.

Lax Monodromy. We wish to obtain integrals of motion via the Lax pair. The latter can be constructed from the parallel transport operator (path-ordered integral, Wilson line) of the Lax connection $A(\lambda)$

$$U(\lambda; t_1, x_1; t_0, x_0) := \mathrm{P}\exp\int_{(t_0,x_0)}^{(t_1,x_1)} A(\lambda). \tag{2.13}$$



Due to flatness it is invariant under continuous deformations of the path contour between (t_0, x_0) and (t_1, x_1). Moreover, shifting the end points amounts to simple differential equations[5]

$$\partial_\alpha^1 U^{10} = A^1 U^{10}, \qquad \partial_\alpha^0 U^{10} = -U^{10} A^0. \tag{2.14}$$

Here the upper indices 0 and 1 represent the points (t_0, x_0) and (t_1, x_1), respectively.

[5]In fact, these equations can be viewed as the defining properties of U^{10}.

The Lax pair is constructed from the parallel transport operator, but we have to take the boundary conditions into account. The simplest choice are periodic boundaries, $\vec{S}(x+L) = \vec{S}(x)$. In this case the Lax pair is defined as

$$L(\lambda) = \mathrm{P} \exp \int_0^L dx\, A_x(\lambda), \qquad M(\lambda) = A_t(\lambda)|_{x=0}. \tag{2.15}$$

The matrix $L(\lambda)$ is also known as the *monodromy matrix*. The above differential equations for U[10] imply the Lax equation[6]

$$\dot{L}(\lambda) = [M(\lambda), L(\lambda)]. \tag{2.16}$$

Integrals of Motion. The eigenvalues or equivalently the traces of powers of L are conserved

$$F_k(\lambda) = \operatorname{tr} L(\lambda)^k. \tag{2.17}$$

One can expand them around some point λ_0, e.g. $\lambda_0 = \infty$, to obtain an infinite tower of conserved quantities

$$F_k(\lambda) = \sum_{r=0}^{\infty} \frac{F_k^{(r)}}{\lambda^r}\,. \tag{2.18}$$

For completeness, we need to show that they are in involution. This follows from a slightly different relationship for Lax matrices with spectral parameters

$$\big\{L_1(\lambda), L_2(\mu)\big\} = \big[r_{12}(\lambda,\mu), L_1(\lambda) \otimes L_2(\mu)\big]. \tag{2.19}$$

with the parameter-dependent classical r-matrix

$$r_{12}(\lambda,\mu) = \frac{\sigma^k \otimes \sigma^k}{2(\lambda-\mu)}\,. \tag{2.20}$$

The latter satisfies the classical Yang–Baxter equation[7]

$$\begin{aligned} &\big[r_{12}(\lambda-\mu), r_{13}(\lambda-\rho)\big] \\ +\,&\big[r_{12}(\lambda-\mu), r_{23}(\mu-\rho)\big] \\ +\,&\big[r_{13}(\lambda-\rho), r_{23}(\mu-\rho)\big] = 0. \end{aligned} \tag{2.21}$$

The above relations imply that

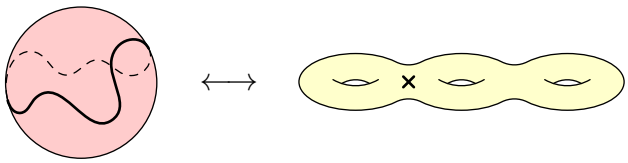
$$\big\{F_k(\lambda), F_l(\mu)\big\} = \big\{F_k^{(r)}, F_l^{(s)}\big\} = 0. \tag{2.22}$$

[6]Here we set $t_0 = t_1 = t$ and $x_0 = 0 \equiv x_1 = L$ so that the time derivative acts on both t_0 and t_1 and $A_t(\lambda)$ is the same at x_0 and x_1 due to periodicity.

[7]This is the simplest form of classical Yang–Baxter equation. There are various modifications for different types of integrable models.

2.2 Spectral Curves

For any solution $\vec{S}(t,x)$ of the equations of motion we know how to compute the monodromy matrix $L(\lambda)$. It contains a lot, perhaps all, information on the integrals of motion. Let us therefore investigate $L(\lambda)$. In particular, the dependence on the spectral parameter $\lambda \in \mathbb{C}$ reveals many properties of the solution by means of the so-called spectral curve. This information is very useful because it allows to construct suitable spectral curves from scratch and thus learn about the conserved charges of a solution without constructing the latter. One can even reconstruct the solution from the spectral curve with some additional dynamical data.



(2.23)

Riemann Sheets. The eigenvalues $\omega_{1,2}(\lambda)$ of $L(\lambda)$ constitute integrals of motion. They are related to the traces of powers of $L(\lambda)$ as follows

$$F_1(\lambda) = \omega_1(\lambda) + \omega_2(\lambda), \qquad F_2(\lambda) = \omega_1(\lambda)^2 + \omega_2(\lambda)^2. \tag{2.24}$$

The inverse relationship reads

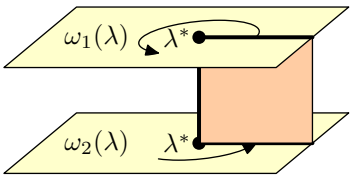
$$\omega_{1,2}(\lambda) = \tfrac{1}{2}F_1(\lambda) \pm \sqrt{\tfrac{1}{2}F_2(\lambda) - \tfrac{1}{4}F_1(\lambda)^2}\,. \tag{2.25}$$

By construction, the functions $F_k(\lambda)$ are analytic (holomorphic) in λ, except for the special point $\lambda = 0$. The same therefore holds for the eigenvalue functions $\omega_{1,2}(\lambda)$ almost everywhere except for square root branch points λ_k^* of the above relations. At these points the two eigenvalues coincide,

$$\omega_1(\lambda_k^*) = \omega_2(\lambda_k^*). \tag{2.26}$$

Square root singularities are special points for a complex function because they have a full angle of rotation of 4π. A rotation of 2π around these points interchanges ω_1 and ω_2. There is nothing wrong with such a behaviour because the two eigenvalues are equivalent by all means. A rotation by 2π merely changes our labelling of the eigenvalues which is inconsequential.

The eigenvalue function $\omega(\lambda)$ can thus be viewed as a function on a two-fold cover of the complex plane, with ω_1 and ω_2 being the two Riemann sheets of the function. Importantly, the sheets are joined along branch cuts which connect the branch points λ_k^* in pairs.



(2.27)

The number and locations λ_k^* of the branch points depends on the underlying solution $\vec{S}(t,x)$ in a very non-trivial fashion. Conversely, the locations of the branch cuts determine the functions $\omega(\lambda)$ as we shall show later. Therefore they determine many (if not all) of the integrals of motion and classify solutions $\vec{S}(t,x)$.

Let us determine further properties of the function $\omega(\lambda)$.

Singularity at $\boldsymbol{\lambda = 0}$. Recall that the monodromy matrix $L(\lambda)$ was constructed by means of the matrix

$$A_x(\lambda) = -\frac{i}{\lambda}\,\vec{\sigma}\cdot\vec{S}. \tag{2.28}$$

It has a pole at $\lambda = 0$ which leads to an essential singularity in $L(\lambda)$. We would like to understand the nature of this singularity better.

To that end, we should diagonalise the connection $A_x(\lambda,x)$ at all x by means of a suitable rotation $U(\lambda,x)$

$$\partial_x + A'_x(\lambda,x) = U(\lambda,x)^{-1}\big(\partial_x + A_x(\lambda,x)\big)U(\lambda,x). \tag{2.29}$$

The rotation matrix is determined such that $A'_x \sim \sigma^3$ is diagonal. This rotation matrix $U(\lambda,x)$ can be perfectly regular at $\lambda = 0$. To leading order in λ a diagonal A'_x is achieved by a rotation which satisfies

$$U(0,x)^{-1}\big(\vec{\sigma}\cdot\vec{S}(x)\big)U(0,x) = \sigma^3\big|\vec{S}(x)\big| = \sigma^3. \tag{2.30}$$

The resulting connection reads

$$\partial_x + A'_x(\lambda,x) = -\frac{i}{\lambda}\,\sigma^3 + \mathcal{O}(\lambda^0) \tag{2.31}$$

and the transformed monodromy matrix is now computed as a plain integral without path ordering

$$L'(\lambda) = \exp\int_0^L dx\, A'_x(\lambda,x). \tag{2.32}$$

Since the monodromy matrices are related by a plain similarity transformation

$$L(\lambda) = U(\lambda,L)^{-1}L(\lambda)U(\lambda,0) = U(\lambda,0)^{-1}L(\lambda)U(\lambda,0), \tag{2.33}$$

we can now easily read off the singular behaviour of the eigenvalues at $\lambda = 0$

$$\omega_{1,2}(\lambda) = \exp\left(\pm\frac{iL}{\lambda} + \mathcal{O}(\lambda^0)\right). \tag{2.34}$$

The higher orders at $\lambda = 0$ can be obtained by a careful analysis involving a λ-dependent rotation $U(\lambda,x)$. This is somewhat laborious, and we shall skip the analysis. Importantly, the resulting conserved charges are *local integrals of motion*. The property of locality is closely related to the pole singularity in $A_x(\lambda)$. In our case the lowest few charges are:

- the total momentum P at $\mathcal{O}(\lambda^0)$,
- the total energy E at $\mathcal{O}(\lambda^1)$,
- higher local charges Q_k involving more than two spatial derivatives at $\mathcal{O}(\lambda^{k-1})$.

Quasi-Momentum and Spectral Curve. For a later reconstruction of the function $\omega(\lambda)$ the existence of essential singularities is inconvenient. They can be removed by considering the logarithm of the function $\omega(\lambda)$ which is known as the *quasi-momentum* $q(\lambda)$

$$q(\lambda) := -i \log \omega(\lambda). \tag{2.35}$$

Evidently, the quasi-momentum has single poles at $\lambda = 0$ with residue $\pm L$.

Note that the quasi-momentum $q(\lambda)$ has inherited the ambiguity of the complex logarithm, and is therefore defined only modulo shifts of 2π. Evidently, one will choose the function to be analytic almost everywhere, but in addition to switching sheets at the existing branch cuts of $\omega(\lambda)$, it can jump by multiples of 2π

$$q_1 \leftrightarrow q_2 + 2\pi n. \tag{2.36}$$

The characteristic number n is constant along the branch cut.

To get rid of these ambiguities, it makes sense to consider the derivative of the quasi-momentum q' or dq as a differential form,

$$q'(\lambda) = -i\omega'(\lambda)/\omega(\lambda). \tag{2.37}$$

This function has only two sheets and algebraic type singularities. It can be viewed as a complex curve, the so-called *spectral curve*. It is therefore ideally suited for complex analysis and for construction purposes.

Note that the curve has inherited some properties from its construction via $\omega(\lambda)$. Let us list them:

- All closed periods of $dq(\lambda)$ on the Riemann surface must be multiples of 2π due to its definition as a logarithmic derivative

$$\oint dq = \oint d\lambda\, q'(\lambda) \in 2\pi\mathbb{Z}. \tag{2.38}$$

- Any point-like singularities cannot have a residue, i.e. they must be poles of higher degree. A pole with a residue requires $q(\lambda)$ to have a logarithmic singularity and thus $\omega(\lambda)$ to have a pole or a zero. This is in conflict with the group nature of the monodromy $L(\lambda)$.
- There is a double pole at $\lambda = 0$ without a residue for the single pole

$$q'_{1,2}(\lambda) = \mp\frac{L}{\lambda^2} + \frac{0}{\lambda} + \dots . \tag{2.39}$$

- The function $q'(\lambda)$ has branch cuts which end in inverse square root branch points.

Special Properties. The matrix $L(\lambda)$ has a further special property which follows from a property of $A_x(\lambda)$ and which influences the behaviour of $\omega(\lambda)$ and $q(\lambda)$.

We know that $A_x \sim \vec{\sigma} \cdot \vec{S}$ in a traceless matrix. After integration and exponentiation we derive

$$\det L(\lambda) = 1, \qquad \omega_1(\lambda)\,\omega_2(\lambda) = 1, \qquad F_2 = F_1^2 - 2. \tag{2.40}$$

For the quasi-momentum it implies that the two sheets differ merely by their sign and potentially by a shift of a multiple of 2π. In order to fix the shift ambiguity on one sheet, we can define the second sheet to be the negative of the first sheet without a shift

$$q_2 = -q_1. \tag{2.41}$$

Passing through a branch cut therefore must include a potential shift by 2π

$$q \leftrightarrow -q + 2\pi n. \tag{2.42}$$

Expansion at $\boldsymbol{\lambda = \infty}$. Another distinguished point is $\lambda = \infty$ where $A_x(\lambda)$ vanishes. The expansion of $L(\lambda)$ is therefore straight-forwardly the expansion of the exponential

$$L(\lambda) = \exp\left(-\frac{i}{\lambda}\, s\vec{\sigma} \cdot \vec{Q} + \mathcal{O}(1/\lambda^2)\right), \tag{2.43}$$

where $\vec{Q}$ is the Noether charge for rotations

$$\vec{Q} = \int_0^L dx\, \vec{J}_t, \qquad \vec{J}_t = \vec{S}. \tag{2.44}$$

For the quasi-momentum it implies

$$q(\lambda) = \pm\frac{1}{\lambda}\,|\vec{Q}|. \tag{2.45}$$

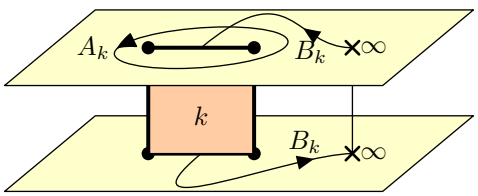
Here we have used and fixed the freedom to shift by multiples of 2π by setting $q(\infty) = 0$.

As an aside, the higher powers of $1/\lambda$ in $L(\lambda)$ correspond to multi-local conserved charges such as

$$\int_0^L dx \int_0^x dx'\, \vec{S}(x) \times \vec{S}(x'). \tag{2.46}$$

Periods and Moduli. The locations λ_k^* of the branch points determine the spectral curve, but they are not immediately telling much about the physical properties of the underlying solution. There are other quantities which are much more suitable: *periods*.

We know that the periods of dq are integer multiples of 2π. To be more concrete, we choose a convenient basis of cycles on the Riemann surface: There is a cycle around each branch cut, these are called the *A-cycles* A_k. Furthermore there is a cycle through each cut, these are called the *B-cycles* B_k.



(2.47)

In this assignment, the distinguished point at $\lambda = \infty$ can be viewed as an infinitesimally short cut. It serves as the distinguished cut which does not contribute to the counting of cycles because

- the combination of all A-cycles combines to an inverse cycle around the remaining singular point $\lambda = \infty$ and
- all B-cycles close though the "cut" at $\lambda = \infty$.

All A-periods vanish while the B-periods yield integers[8]

$$\oint_{A_k} dq = 0, \qquad \int_{B_k} dq = 2\pi n_k. \tag{2.48}$$

The integers n_k describe the jump of the quasi-momentum $q(\lambda)$ at the branch cuts. They are called *mode numbers*.

We can measure another characteristic number for each branch cut as the A-period of $\lambda\, dq$, the so-called *filling* K_k

$$K_k = \frac{1}{2\pi i} \oint_{A_k} \lambda\, dq. \tag{2.49}$$

It is a measure of the length of the branch cut, and unlike n_k it takes continuous values. Note that quantisation of the classical theory renders these numbers to be quantised as integers, too.

Finite Gap Construction. Let us summarise the properties of the spectral curve $q'(\lambda)$:

- The function has two Riemann sheets, it is single-valued on the Riemann surface, the sum of the Riemann sheets is zero.
- The function has branch points λ_k^* of the type $1/\sqrt{\lambda - \lambda_k^*}$.
- There is a fixed pole $\pm L/\lambda^2 + 0/\lambda$ at $\lambda = 0$.
- The asymptotic behaviour at $\lambda \to 0$ is $\sim 1/\lambda^2$.
-

For spectral curves with finitely many cuts ("finite gap") we can make a general ansatz as an algebraic curve

$$q'(\lambda) = \pm \frac{P_N(\lambda)}{\lambda^2 \sqrt{Q_N(\lambda)}}\,, \tag{2.50}$$

where P_N and Q_{2N} are polynomials of degree N and $2N$, respectively, with $2N + 2$ free parameters in total. This ansatz automatically satisfies several of the above properties, the remaining properties constrain some of the parameters as follows:

- N A-periods $\oint dq = 0$,
- N B-periods $\int dq = 2\pi n_k$,
- N fillings $\oint \lambda dq \sim K_k$,

[8] For integer periods one can always make at least half of them vanish by a suitable choice of independent cycles and thus of Riemann sheets and cuts.

- 1 coefficient of the $1/\lambda^2$ pole at $\lambda = 0$,
- 1 ambiguity of overall rescaling of P and $\sqrt{Q}$.

We learn that all degrees of freedom are fixed by the knowledge of the (discrete) mode numbers n_k and the (continuous) fillings K_k. All integrals of motion (momentum, energy, spin, higher charges) follow from this finite gap solution.

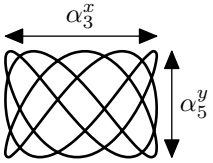
This classifies solutions with finite genus N. One could view the more general spectral curves with infinitely many cuts as the limiting case $N \to \infty$.

Physics of Spectral Curves. Finite genus spectral curves are specified by one discrete mode number n_k and one continuous filling K_k for each cut. This matches qualitatively with the spectrum for $(1+1)$-dimensional field theories with closed boundary conditions, such as string theory. Let us discuss the latter:

For solutions $\vec{S}$ near a trivial vacuum solution $\vec{S}_0$ one could make an ansatz in terms of Fourier modes

$$\vec{S} = \vec{S}_0 + \sum_n \vec{\alpha}_n \exp(2\pi i n x/L). \tag{2.51}$$

Here, the mode numbers n are discrete whereas the amplitudes $\vec{\alpha}_n$ are continuous.



(2.52)

Finite gap solutions represent solutions where only a finite number of Fourier modes n_k are active, $K_k \sim |\alpha_k|^2 > 0$. Note that for a non-linear problem[9] the Fourier mode expansion leads to complicated non-linear relationships of the α's. The spectral curve automatically takes care of this complication. It can be viewed as a non-linear version of the Fourier transformation by means of complex analysis which is perfectly adapted to our physics model.

[9] In our model, the constraint $\vec{S}^2 = 1$ is responsible for non-linearity.

3 Integrable Spin Chains

We now proceed to integrable quantum mechanical models. They are instructive because:

- they form a large class of integrable models,
- they can be treated uniformly,
- they have many parameters to tune,
- short chains are genuine quantum mechanical models,
- long chains approximate $(1+1)$D quantum field theories,
- for large quantum numbers they are approximated by classical models,
- they model magnetic materials.

Magnets. Ansatz: A magnetic material consists of many microscopic magnets, e.g. atoms with spin. The energy of the material depends on the configuration of nearby spins.

nearby spins	ferromagnet	anti-ferromagnet
opposite alignment $\uparrow\downarrow$	high energy	low energy
equal alignment $\uparrow\uparrow$	low energy	high energy

(3.1)

Two well known models of magnets are:

- Ising model, a model of statistical mechanics. It consists of a lattice of spins taking values $\uparrow$, $\downarrow$. The alignment of nearest neighbours determines the energy.
- Heisenberg chain, a quantum mechanical model. It consists of a chain of spin states $|\uparrow\rangle$, $|\downarrow\rangle$. The Hamiltonian acts on nearest neighbours.

In the following we shall discuss the Heisenberg spin chain in detail.

3.1 Heisenberg Spin Chain

Let us start by introducing the model and investigating its spectrum.

Setup. A single spin state can be $|\downarrow\rangle$ or $|\uparrow\rangle$ or any complex linear combination of these two. In other words, a spin is described by an element of the vector space

$$\mathbb{V} = \mathbb{C}^2. \tag{3.2}$$

A spin chain of length L is the L-fold tensor product

$$\mathbb{V}^{\otimes L} = \mathbb{V}_1 \otimes \ldots \otimes \mathbb{V}_L. \tag{3.3}$$

This space serves as the Hilbert space of our model. It has finite dimension 2^L. A basis is given by the "pure" states, e.g.

$$|\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\uparrow\downarrow\rangle. \tag{3.4}$$

The Hamiltonian operator $H : \mathbb{V}^{\otimes L} \to \mathbb{V}^{\otimes L}$ is homogeneous and acts on nearest neighbours

$$\mathcal{H} = \sum\nolimits_k \mathcal{H}_{k,k+1}, \qquad \mathcal{H}_{k,l} : \mathbb{V}_k \otimes \mathbb{V}_l \to \mathbb{V}_k \otimes \mathbb{V}_l. \tag{3.5}$$

The pairwise kernel $H_{k,l}$ for the Heisenberg chain reads

$$\mathcal{H}_{k,l} = \lambda_0(1 \otimes 1) + \lambda_x(\sigma^x \otimes \sigma^x) + \lambda_y(\sigma^y \otimes \sigma^y) + \lambda_z(\sigma^z \otimes \sigma^z). \tag{3.6}$$

It is integrable for all values of the coupling constants $\lambda_0, \lambda_x, \lambda_y, \lambda_z$. Several useful cases can be distinguished:

- The most general (and most complicated) case is $\lambda_x \neq \lambda_y \neq \lambda_z \neq \lambda_x$: This is the so-called "XYZ" model.
- Many simplifications occur for $\lambda_x = \lambda_y \neq \lambda_z$: This is the so-called "XXZ" model.
- Symmetry is enhanced for $\lambda_x = \lambda_y = \lambda_z$: This is the so-called "XXX" model.

We shall mainly use the XXX model with the choice[1]

$$\lambda_0 = -\lambda_x = -\lambda_y = -\lambda_z = \tfrac{1}{2}\lambda. \tag{3.7}$$

With this choice the Hamiltonian kernel reads

$$\mathcal{H}_{k,l} = \lambda(\mathcal{I}_{k,l} - \mathcal{P}_{k,l}), \tag{3.8}$$

where $\mathcal{I}_{k,l}$ is the identity operator and $\mathcal{P}_{k,l}$ the permutation on the two equivalent spaces $\mathbb{V}_k$ and $\mathbb{V}_l$. Note that $\lambda > 0$ implies ferromagnetic behaviour whereas $\lambda < 0$ implies anti-ferromagnetic behaviour.[2]

Boundary Conditions. To complete the definition of the model, we must specify the boundary conditions. Typical choices are

- open chain:

$$\mathcal{H} = \sum_{k=1}^{L-1} \mathcal{H}_{k,k+1}, \tag{3.9}$$

- closed chain: identify sites periodically such that $\mathbb{V}_{L+1} = \mathbb{V}_1$

$$\mathcal{H} = \sum_{k=1}^{L} \mathcal{H}_{k,k+1}, \tag{3.10}$$

- infinite chain:

$$\mathcal{H} = \sum_{k=-\infty}^{+\infty} \mathcal{H}_{k,k+1}. \tag{3.11}$$

[1] The value of λ_0 is largely irrelevant because it merely induces an overall shift of all energies. Our choice sets the energy of a reference state to zero.

[2] We shall be interested in all states of the model, hence the difference between the ferromagnetic and anti-ferromagnetic case is merely an overall sign of the energy spectrum. The distinction between the two cases becomes relevant only when considering the ground state and its low-energy excitations.

Other choices that are sometimes encountered include:

- twists of the closed boundary conditions,
- open boundary conditions with specific boundary Hamiltonians,
- semi-infinite chains.

Some of these boundary conditions are compatible with integrability, others may not.

Boundary conditions have a strong impact on the spectrum: Infinite chains generally have a continuous spectrum while finite chains by have a discrete spectrum by definition. This makes the spectral problem more interesting for finite chains. Here, the closed chains are typically easier to handle than open chains, therefore we shall mainly consider the former.

Spectrum. Consider a finite chain, how to obtain the spectrum?

- Enumerate a basis of $\mathbb{V}^{\otimes L}$, e.g. $|\downarrow\ldots\downarrow\downarrow\rangle$, $|\downarrow\ldots\downarrow\uparrow\rangle$, ... amounting to 2^L states in total.
- Evaluate $\mathcal{H}$ in this basis as a $2^L \times 2^L$ matrix. This uninspiring task of basic combinatorics leads to a sparse matrix of integer entries.
- Next solve the eigenvalue problem of the Hamiltonian matrix.

The problem is ideally suited for computer algebra:

- One can automatically evaluate the Hamiltonian as a matrix for fairly large L.
- An exact diagonalisation in terms of algebraic numbers if feasible only for small L.
- Numerical evaluation of the eigenvalues allows slightly larger values of L.
- The spectrum is a big mess.
- Eigenvalues appear in multiplets.

Spectrum in Mathematica. Let us present a concise implementation of the XXX Hamiltonian in `Mathematica`.

First, we need to find a way to represent spin chain states. An immediate thought would be to define them as vectors with 2^L components. A drawback of this approach is that one obtains rather abstract and obscure objects which grow exponentially fast with L and which are not so easy to act upon. An alternative and more symbolic approach is to "define" a set of abstract basis vectors and allow for linear combinations. For example, we can represent pure spin chain states by functions whose arguments denote the spin orientations

$$|\uparrow,\uparrow,\downarrow,\uparrow,\downarrow\rangle \to \texttt{State[1,1,0,1,0]}. \tag{3.12}$$

The function `State` is undefined by default, so it remains unevaluated and can be used to represent linear combinations, e.g.

$$10\,\texttt{State[1,1,0,1,0]} - 5\,\texttt{State[1,0,1,1,0]}. \tag{3.13}$$

Next we have to represent the Hamiltonian $\mathcal{H}$ through some replacement operator

$$\texttt{Ham} : \sum *\texttt{State[...]} \to \sum *\texttt{State[...]}. \tag{3.14}$$

A homogeneous nearest neighbour Hamiltonian can be implemented by the following code:

```
Ham[X_] :=
  X /. Psi_State :> Module[{k, L=Length[Psi]},
    Sum[HamAt[Psi, k, Mod[k+1, L, 1]],
      {k, L}]];
```
(3.15)

This function replaces (`/.`, `ReplaceAll`) every occurrence of `State` in the argument `X` with the homogeneous action of the kernel `HamAt`. Some notes:

- `Psi_State` symbolises any object `State[...]`, i.e. any object with head `State`.[3]
- The use of the replacement operator `:>` (`RuleDelayed`) as opposed to `->` (`Rule`) is essential because it evaluates the right hand side only after insertion of `Psi`.
- The above definition assumes that the argument `X` is a linear combination of `State` objects. If `X` is not a linear combination of `State` objects, `Ham` does whatever it does (replace objects). Lists, vectors, matrices, nested lists of linear combinations of `State` objects are permissible as arguments: `Ham` will act on each element individually.
- The construct `Module` defines a local variable `k` [4] and a local variable `L` assigned with the length of the state `Psi`.

The Hamiltonian kernel for the XXX model can be defined as

```
HamAt[Psi_State, k_, l_] :=
  Psi - Permute[Psi, Cycles[{{k,l}}]];
```
(3.16)

It uses some pre-defined combinatorial methods to implement the permutation of two sites in the symbol `Psi`.

We are now ready to act on states. In order to obtain the complete spectrum we have to enumerate a basis of $\mathbb{V}^{\otimes L}$. As a shortcut, we can employ the binary representation of integers $0, \ldots, 2^L - 1$:

```
Basis[L_] :=
  Table[State @@ IntegerDigits[k, 2, L],
    {k, 0, 2^L-1}];
```
(3.17)

Here the operator `@@` (`Apply`) replaces the head of the binary representation of `k` (which is `List`) with `State`. The variable `states` is now a list of pure basis states.

To evaluate the Hamiltonian on the states we can use the following construct:

```
HamMat[states_] :=
  Module[{X=Ham[states]},
    Coefficient[X, #] & /@ states];
```
(3.18)

Some notes:

[3] Almost all objects in `Mathematica` (except variables and concrete numbers) are *headed lists*. They can be treated much like lists (which are in fact objects with head `List`).

[4] Sometimes using the same variable names as arguments of a `Sum` and elsewhere can lead to undesired interference (depending on the order of evaluation of sub-expressions). To avoid a potential interference it makes sense to make the summation variable local.

- `Ham[states]` evaluates `Ham` on every element of the list `states`. Usually, one would have to explicitly declare this behaviour for the function `Ham` by means of `SetAttributes[Ham, Listable]`. In our case, the definition via a replacement rule automatically implements this desired behaviour.
- The operator `&` (`Function`) represents a *pure function* (a function without a declaration) which returns `Coefficient[X, #]` where `#` is the argument passed to the function. In practice it extracts the coefficient of the argument within `X`.
- The operator `/@` (`Map`) evaluates the above pure function on all elements of the list `states`. This is the matrix representation of `Ham` in the basis `states`.[5]

To finally extract the eigenvalues, generate the Hamiltonian matrix via (remember to substitute or define `L` as a not too large positive integer)

```
emat = HamMat[Basis[L]];
```
(3.19)

and use `Eigenvalues[emat]`, `Eigenvalues[N[emat]]` or `Eigenvalues[N[emat, 20]]`.

Symmetry. The XXX Hamiltonian has a SU(2) Lie group symmetry because the kernel $\mathcal{H}_{k,l}$ is formulated as a manifestly SU(2) invariant operator.

We can set up a representation $\mathcal{J}^\alpha$, $\alpha = x, y, z$, of the Lie algebra $\mathfrak{su}(2)$ on spin chains

$$\mathcal{J}^\alpha = \sum_{k=1}^{L} \sigma_k^\alpha. \tag{3.20}$$

- This is a tensor product representation of L spin 1/2 irreps of $\mathfrak{su}(2)$ given by the Pauli matrices σ_k^α acting on site k.
- The Hamiltonian is invariant
$$[\mathcal{J}^\alpha, \mathcal{H}] = 0. \tag{3.21}$$
- The tensor product is decomposable, for the shortest few chains one finds by the well-known tensor product rules for $\mathfrak{su}(2)$:

$$\begin{array}{ll} L=2: & (1)+\ (0); \\ L=3: & (\frac{3}{2})+2(\frac{1}{2}); \\ L=4: & (2)+3(1)+2(0); \\ L=5: & (\frac{5}{2})+4(\frac{3}{2})+5(\frac{1}{2}); \\ L=6: & (3)+5(2)+9(1)+5(0); \\ & \ldots . \end{array} \tag{3.22}$$

  Here (s) denotes a finite irrep of spin s.
- Each multiplet has one common eigenvalue.

The spectrum of the closed chain of small length L in units of λ takes a fairly

[5] Potentially, one should `Transpose` the matrix.

simple form:

L	eigenvalue multiplets		
2	$(1)\times 0,$	$(0)\times 4;$	
3	$(\frac{3}{2})\times 0,$	$2(\frac{1}{2})\times 3;$	
4	$(2)\times 0,$	$2(1)\times 2,$	$(1)\times 4,$
	$(0)\times 6,$	$(0)\times 2;$	
5	$(\frac{5}{2})\times 0,$	$2(\frac{3}{2})\times \frac{1}{2}(5+\sqrt{5}),$	$2(\frac{3}{2})\times \frac{1}{2}(5-\sqrt{5}),$
	$(\frac{1}{2})\times 4,$	$2(\frac{1}{2})\times 4+\sqrt{5},$	$2(\frac{1}{2})\times 4-\sqrt{5};$
6	$(3)\times 0,$	$2(2)\times 3,$	$2(2)\times 1,$
	$(2)\times 4,$	$2(1)\times \frac{1}{2}(7+\sqrt{17}),$	$2(1)\times \frac{1}{2}(7-\sqrt{17}),$
	$2(1)\times 5,$	$(1)\times 5+\sqrt{5},$	$(1)\times 5-\sqrt{5},$
	$(1)\times 2,$	$(0)\times 5+\sqrt{13},$	$(0)\times 5-\sqrt{13},$
	$2(0)\times 4,$	$(0)\times 6;$	
	....		

(3.23)

Note that the $\mathfrak{su}(2)$ eigenvalue multiplets typically appear with an extra multiplicity of 1 or 2. The pairing is largely a consequence of parity symmetry. However, parity is not sufficient to explain all of the pairings. Such extra pairings can be related to integrability.

3.2 Coordinate Bethe Ansatz

The Heisenberg spin chain is quantum integrable. Unfortunately, there is no universal notion of quantum integrability as in the finite-dimensional classical case (Liouville). In particular, it is unclear how to define the number of degrees of freedom in a quantum theory. Let us therefore inspect the consequences of integrability:

Closed Chain Bethe Equations. Consider a set of M algebraic equations (Bethe equations) for the M variables $u_k \in \mathbb{C}$ (Bethe roots)

$$\left(\frac{u_k+\frac{i}{2}}{u_k-\frac{i}{2}}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{M} \frac{u_k-u_j+i}{u_k-u_j-i} \qquad \text{for } k=1,\dots,M. \tag{3.24}$$

Claim: for every eigenstate of $\mathcal{H}$ there is a solution of the above equations with $M \leq L/2$ distinct Bethe roots u_k.[6] The energy eigenvalue of this state can be read off easily

$$E = \lambda \sum_{k=1}^{M} \left(\frac{i}{u_k+\frac{i}{2}} - \frac{i}{u_k-\frac{i}{2}}\right). \tag{3.25}$$

[6]There are some subtleties related to the SU(2) symmetry for the XXX model, and one has to pay attention to Bethe roots at ∞, $\pm\frac{i}{2}$ and Bethe roots separated by i.

Example: $L = 6$, $M = 3$ (corresponding to a spin singlet)

$$u_{1,2} = \pm\sqrt{-\frac{5}{12} + \frac{\sqrt{13}}{6}}\,, \quad u_3 = 0, \qquad E = \lambda(5 + \sqrt{13}). \tag{3.26}$$

Benefits:

- We have transformed a problem of linear algebra directly to algebraic equations. We can thus skip combinatorics and characteristic polynomials.
- We can use the Bethe equations efficiently for approximations at large L and M. For example, the anti-ferromagnetic ground state can be approximated at large L in which case the Bethe equations turn into integral equations.

In the following we shall derive the Bethe equations from the coordinate Bethe ansatz.

Vacuum State. Start with a simple state, the *ferromagnetic vacuum*

$$|0\rangle = |\downarrow\downarrow\ldots\downarrow\rangle. \tag{3.27}$$

By construction this state has zero energy

$$\mathcal{H}_{k,k+1}|0\rangle = \mathcal{I}_{k,k+1}|0\rangle - \mathcal{P}_{k,k+1}|0\rangle = |0\rangle - |0\rangle = 0. \tag{3.28}$$

Therefore $\mathcal{H}|0\rangle = 0$ and the ground state energy is zero

$$E = 0. \tag{3.29}$$

This solves the problem for $M = 0$ corresponding to the multiplet $(L/2)$.

Magnon States. Now flip one spin at site k

$$|k\rangle = |\downarrow\ldots\downarrow\overset{k}{\uparrow}\downarrow\ldots\downarrow\rangle. \tag{3.30}$$

These states enumerated by k form a closed sector under the Hamiltonian due to conservation of the z-component of spin $\mathcal{J}^z$.

How to obtain eigenstates of $\mathcal{H}$? Note that the Hamiltonian is homogeneous and commutes with a shift of the chain by one unit.[7] We can thus look for simultaneous eigenstates of the Hamiltonian and the shift operator. Momentum eigenstates are plane waves[8]

$$|p\rangle = \sum\nolimits_k e^{ipk}|k\rangle. \tag{3.31}$$

This state is called a *magnon state*. It can be viewed as a particle excitation[9] of the above vacuum state.

[7] The lattice shift is a discrete version of the momentum generator.

[8] The notation is slightly ambiguous, but it should become clear from the context whether $|*\rangle$ refers to a position eigenstate $|k\rangle$ or a momentum eigenstate $|p\rangle$.

[9] Here the notion of particle is an object which carries an individual momentum p.

Since there is a unique state with a given momentum p, it must already by an energy eigenstate. We can now act with $\mathcal{H}$ on $|p\rangle$ and obtain (after a shift of summation variable to match the states on the r.h.s.)

$$\begin{aligned}\mathcal{H}|p\rangle &= \lambda \sum\nolimits_k e^{ipk}\Big(\overbrace{|k\rangle - |k-1\rangle}^{H_{k-1,k}} + \overbrace{|k\rangle - |k+1\rangle}^{H_{k,k+1}}\Big) \\ &= \lambda \sum\nolimits_k e^{ipk}\big(1 - e^{ip} + 1 - e^{-ip}\big)|k\rangle \\ &= e(p)|p\rangle \end{aligned} \tag{3.32}$$

with the *magnon dispersion relation*

$$e(p) = 2\lambda(1-\cos p) = 4\lambda \sin^2(\tfrac{1}{2}p). \tag{3.33}$$

For a closed chain, the momentum is quantised by the periodic boundary conditions to

$$p = \frac{2\pi i n}{L}\,, \qquad \text{where } n = 0, \ldots, L-1. \tag{3.34}$$

For an infinite chain p is a continuous parameter. Note that in both cases the momentum is defined only modulo 2π because the position k is sampled only at the discrete lattice positions. A shift by 2π corresponds to a change of Brillouin zone which leaved the eigenstate unchanged.

This solves the problem for $M = 1$ corresponding to the multiplets $(L/2-1)$.[10]

Scattering Factor. We continue with states with two spin flips

$$|k<l\rangle = |\downarrow \ldots \downarrow \overset{k}{\uparrow} \downarrow \ldots \downarrow \overset{l}{\uparrow} \downarrow \ldots \downarrow\rangle. \tag{3.35}$$

Here we make the assumption that $k < l$. Again, these states form a closed sector for the Hamiltonian, and we wish to construct eigenstates.

When the spin flips are well-separated we can treat the state as the combination of two individual magnons. The nearest neighbour Hamiltonian will hardly every see both spin flips at the same time, therefore we can make an ansatz for eigenstates of the form

$$|p<q\rangle = \sum_{k<l=-\infty}^{+\infty} e^{ipk+iql}|k<l\rangle. \tag{3.36}$$

Some comments:

- This state has overall momentum $P = p + q$.
- The momenta of the individual spin flips are not well-defined because their wave functions do not extend over the whole chain, but are constrained by an ordering of the spin flips. Nevertheless we can use p and q as labels for a particular state.
- The notation $|p<q\rangle$ is not meant to imply that p is less than q, but rather that the magnon with momentum p *is to the right of* the magnon with momentum q.

[10] The state with $p = 0$ belongs to the multiplet $(L/2)$ discussed in the context of the vacuum.

- For the time being we shall only consider an infinite chain so that we do not have to worry about boundary conditions.

By construction, each of these states is almost an eigenstate with eigenvalue

$$E = e(p) + e(q). \tag{3.37}$$

Acting with the combination $\mathcal{H} - e(p) - e(q)$ on $|p < q\rangle$ yields

$$\left(e^{ip+iq} - 2e^{iq} + 1\right) \sum_{k=-\infty}^{+\infty} e^{i(p+q)k} |k < k+1\rangle. \tag{3.38}$$

Only a contact term $\sum_k e^{i(p+q)k}|k < k+1\rangle$ remains and violates the eigenstate condition. Since this state is symmetric in p and q we can act on the state $|q < p\rangle$ with the magnon momenta interchanged and obtain a proportional term

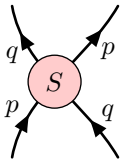
$$\left(e^{ip+iq} - 2e^{ip} + 1\right) \sum_{k=-\infty}^{+\infty} e^{i(p+q)k} |k < k+1\rangle. \tag{3.39}$$

We can now patch together the two partial wave functions and construct an exact eigenstate[11]

$$|p, q\rangle = |p < q\rangle + S(p, q)|q < p\rangle \tag{3.40}$$

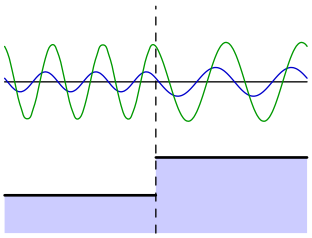
with a *scattering factor* S

$$S(p, q) = -\frac{e^{ip+iq} - 2e^{iq} + 1}{e^{ip+iq} - 2e^{ip} + 1}\,. \tag{3.41}$$



The scattering factor is analogous to the scattering factor or scattering matrix in QM and QFT.[12]

The process of patching together two wave functions is analogous to the construction of quantum mechanical wave functions at a potential step.



(3.42)

In our context, the distance of the magnons is the relevant position variable and the potential step is at the minimum distance of 1 lattice site.

[11] We pay no attention to the overall normalisation of the state. Therefore the state $|p, q\rangle$ is independent of the order of p and q (up to normalisation).

[12] The setup for these two objects is slightly different: The ordinary scattering factor relates ingoing and outgoing states where each particle has a well-defined momentum in the distant past and distant future. The scattering factor of the Bethe ansatz relates two partial wave functions with different ordering of the individual particles. The latter is something that works only with one spatial dimension. Therefore an ordinary scattering factor corresponds to a time-like process whereas the scattering factor in the Bethe ansatz corresponds to a space-like process.

Factorised Scattering. Before considering closed chains, let us have a look at three-magnon states. There are $6 = 3!$ asymptotic regions for magnons which carry one momentum p_k each. A useful ansatz for an eigenstate is the so-called *Bethe ansatz*

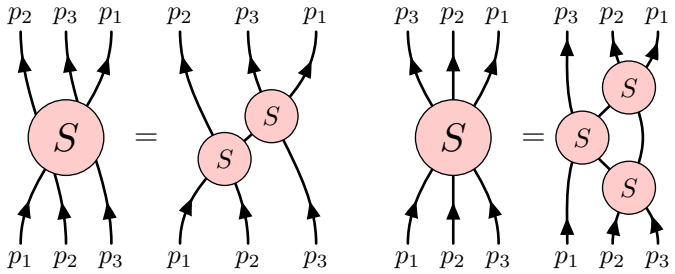
$$\begin{aligned}|p_1, p_2, p_3\rangle = |p_1 < p_2 < p_3\rangle &+ S_{12}S_{13}S_{23}|p_3 < p_2 < p_1\rangle \\ &+ S_{12}|p_2 < p_1 < p_3\rangle + S_{13}S_{23}|p_3 < p_1 < p_2\rangle \\ &+ S_{23}|p_1 < p_3 < p_2\rangle + S_{12}S_{13}|p_2 < p_3 < p_1\rangle. \end{aligned} \tag{3.43}$$

All pairwise contact terms vanish by construction due to the choice of appropriate pairwise scattering factors between any two partial wave functions. There could in principle be a triple contact term

$$\big(\mathcal{H} - E\big)|p_1, p_2, p_3\rangle \sim \sum\nolimits_k e^{iPk}|k < k+1 < k+2\rangle. \tag{3.44}$$

Due to a miracle this contact term is absent without further ado. This miracle is called integrability.[13] It works analogously for any number of magnons as we shall discuss below. We only need the two-magnon scattering factor to construct arbitrary states.

In other words, scattering of more than two magnons factorises into a sequence of pairwise magnon scattering processes, for example:



(3.45)

For pairwise scattering, conservation of total momentum and total energy ensures that the individual momenta are not deformed but merely exchanged. Therefore, in factorised scattering processes, the particle momenta are only ever permuted. Conversely, if the particle momenta are merely permuted by scattering, the above ansatz for the complete wave function must be complete, and scattering factorises.

Factorised scattering means that there is no elementary scattering factor for three or more magnons. Such a factor would lead to a substantially different behaviour in that it must deform the momenta of the particles continuously. This is an option for three or more particles because conservation of total momentum and total energy are not sufficient to guarantee the conservation of all individual momenta. Conservation of additional commuting charges (which can be measured simultaneously) excludes deformations of the kinematical configuration, and therefore implies factorised scattering.

[13] The absence of the contact term is potentially a consequence of the pairwise nature of the Hamiltonian. This is not in contradiction with the fact that most models with nearest neighbour Hamiltonians are non-integrable. Generically these models have a pairwise scattering matrix which is not consistent with the assumption of factorised scattering in which case the above ansatz has leftover pairwise contact terms.

Solution of the Infinite Chain. We have found an explicit and exact solution for the eigenstates of the infinite chain with an arbitrary number of magnons.

$$\begin{aligned} |0\rangle &= |\downarrow\ldots\downarrow\rangle, & E &= 0, \\ |p\rangle &= \sum\nolimits_k e^{ipk}|\ldots\overset{k}{\uparrow}\ldots\rangle, & E &= e(p), \\ |p,q\rangle &= |p<q\rangle + S(p,q)|q<p\rangle, & E &= e(p)+e(q), \\ |p_k\rangle &= \sum_{\pi\in S_M} S_\pi |p_{\pi(1)}<\ldots<p_{\pi(M)}\rangle, & E &= \sum_k e(p_k) \end{aligned} \tag{3.46}$$

The momenta p_k are arbitrary numbers, therefore the spectrum is continuous.

Note:

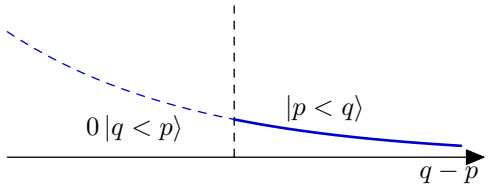
- The ordering of the p_k does not matter: magnons are identical particles.
- The momenta p_k are defined modulo 2π: they move on a lattice.
- The momenta should be real for wave functions to be normalisable (in the ordinary sense of plane waves).
- For two identical momenta

  $$S(p,p) = -1 \quad\Longrightarrow\quad |p,p,\ldots\rangle = 0. \tag{3.47}$$

  The value of the scattering factor and the exclusion principle indicates that the particles obey Fermi statistics. The XXX model on the infinite chain is equivalent to *free fermions* on a one-dimensional lattice!
- Zero-momentum particles are special:

  $$S(p,0) = 1, \qquad e(0) = 0. \tag{3.48}$$

  They behave as free bosons which do not interact with any of the other particles. They represent the $\mathfrak{su}(2)$ ladder operators which allow to walk between the states of a SU(2) multiplet.
- Whenever $S(p,q) = 0,\infty$, some partial wave functions of a state $|p,q,\ldots\rangle$ have coefficient 0.[14] This allows the involved momenta to be complex under certain conditions. For this to happen, the exponentially growing regions of the wave function, which would normally make the state non-normalisable, must be removed by a zero coefficient.



(3.49)

Such states are called *bound states*. They can be viewed as different types of particle excitations with one independent (real) momentum and a different dispersion relation. Bound states made from more than two magnons also exist.

[14] One has to arrange the overall normalisation such that none of the relevant scattering factors is ∞.

Periodicity and Bethe Equations. We know how to solve the infinite XXX chain, but we would like to understand the spectrum of the finite closed chain. To this end, we can compare states of the closed chain with periodic states of the infinite chain. There are at least two conceivable notions of periodic states:

- states with periodic excitations

$$|\ldots, k_1 - L, k_1, k_1 + L, \ldots, k_2 - L, k_2, k_2 + L, \ldots\rangle. \tag{3.50}$$

- states with a periodic wave function

$$\langle k, \ldots |\Psi\rangle = \langle k + L, \ldots |\Psi\rangle. \tag{3.51}$$

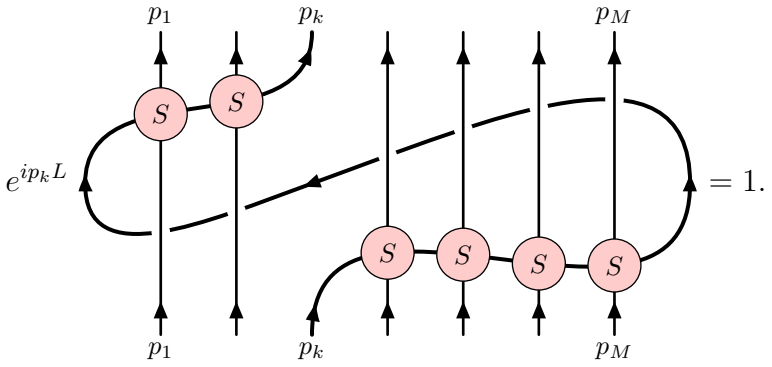
The advantage of the latter point of view is that it requires only finitely many excitations. Let us therefore continue along these lines:

- Consider a position space configuration where the magnons are separated by less than L sites.[15]
- Focus on the leftmost excitation and pay attention to how the wave function of an eigenstate evolves as this excitation is shifted towards the right.
- Moving the excitation by L sites generates a factor of $e^{ip_k L}$ by construction.
- Along the way, it will move past all the other excitations and pick up a factor of $S(p_k, p_j)$ for each permutation.[16]
- The eigenstate is periodic if all the phase factors multiply to 1.
- Such a periodic wave function can be lifted to the wave function on a closed chain.[17]

This leads to the *Bethe equations* for a closed chain

$$e^{ip_k L} \prod_{\substack{j=1 \\ j\neq k}}^{M} S(p_k, p_j) = 1, \quad \text{for all } k = 1, \ldots, M. \tag{3.52}$$

Graphically, the Bethe equations can be represented as follows:



(3.53)

[15] In fact, the states are not actually periodic when the magnons are spread along the chain by more than L sites.

[16] The wave function changes rapidly at a single site, there is no interaction of the magnons at a distance. This is a crucial insight to make this construction exact.

[17] The above restriction to a range of at most L sites is crucial. When moving the magnon further towards the right, no further scattering factor can be picked up, and the wave function does not have a different periodic behaviour.

They amount to one equation for each unknown variable p_k. This effectively quantises the spectrum. A simple consistency requirement for the closed chain already leads to a discrete set of solutions.

The total energy and total momentum of a solution can be read off from the set of magnon momenta p_k

$$E = \sum_{k=1}^{M} e(p_k), \qquad P = \sum_{k=1}^{M} p_k. \tag{3.54}$$

One can simply derive a useful statement on P by multiplying all Bethe equations, namely

$$e^{iPL} = 1. \tag{3.55}$$

This relationship follows from triviality of an overall shift by L sites where e^{iP} is the eigenvalue of the cyclic shift operator.

Rapidities. It is convenient to introduce a different set of variables u_k instead of the momenta p_k.

$$p_k = 2 \operatorname{arccot} 2u_k, \qquad u_k = \tfrac{1}{2} \cot \tfrac{1}{2} p_k. \tag{3.56}$$

The scattering factor simplifies to a rational function

$$S(u,v) = \frac{u - v - i}{u - v + i}\,. \tag{3.57}$$

The Bethe equations then take the form introduced earlier

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L = \prod_{\substack{j=1 \\ j \neq k}}^{M} \frac{u_k - u_j + i}{u_k - u_j - i} \qquad \text{for } k = 1, \ldots, M. \tag{3.58}$$

and the energy and momentum eigenvalues are obtained via

$$e^{iP} = \prod_{k=1}^{M} \frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\,, \qquad E = \lambda \sum_{k=1}^{M} \left(\frac{i}{u_k + \frac{i}{2}} - \frac{i}{u_k - \frac{i}{2}} \right). \tag{3.59}$$

Let us mention some special points and configurations:

- The u_k are real or form complex conjugate pairs.[18]
- All u_k must be distinct except for the special value $u_k = \infty$ which can appear several times.
- The $\mathfrak{su}(2)$ ladder operators at $p_k = 0$ correspond to $u_k = \infty$.
- The special values $u_k = \pm\frac{i}{2}$ where $p_k = \infty$ which typically do not appear in physically relevant solutions. However, some relevant singular solutions exist.
- There is no analog of the periodicity of the p_k for the u_k.
- We should restrict to $M \leq \frac{1}{2}L$; the other states with $M > \frac{1}{2}L$ are represented via a collection of $u_k = \infty$ added to a solution with $M \leq \frac{1}{2}L$.
- Bound states with $S(u_k, u_j) = 0, \infty$ are obtained for the simple condition $u_k = u_j \pm i$. Higher bound states correspond to so-called *Bethe strings* $u_k = u_0 + ik$.

[18] Normalisability is not an issue for finite chains.

3.3 Generalisation

Open Chains. Consider an open chain with Hamiltonian

$$\mathcal{H} = \sum_{k=1}^{L-1} \mathcal{H}_{k,k+1}. \tag{3.60}$$

To quantify the effect of the boundaries, consider a semi-infinite chain starting at site $k = 1$. Act with $\mathcal{H} - e(p)$ on a one-magnon state $|+p\rangle$.

$$\big(\mathcal{H} - e(p)\big)|+p\rangle = (1 - e^{+ip})|1\rangle. \tag{3.61}$$

As for the two-magnon state, there is a residual term located at the boundary. This term can be compensated by another partial eigenstate with equal energy $e(\bar{p}) = e(p)$, namely $\bar{p} = -p$.

$$\big(\mathcal{H} - e(p)\big)|-p\rangle = (1 - e^{-ip})|1\rangle. \tag{3.62}$$

Now combine the states into an exact eigenstate[19] [20]

$$|p\rangle = e^{-ip}|+p\rangle + e^{+ip} K_{\mathrm{L}}(-p)|-p\rangle \tag{3.63}$$

with the boundary scattering factor

$$K_{\mathrm{L}}(-p) = -e^{-2ip}\frac{1 - e^{+ip}}{1 - e^{-ip}} = e^{-ip}. \tag{3.64}$$

Similarly, one can construct exact eigenstates for a semi-infinite chain ending at site $k = L$

$$|p\rangle = e^{-ipL}|+p\rangle + e^{+ipL} K_{\mathrm{R}}(+p)|-p\rangle \tag{3.65}$$

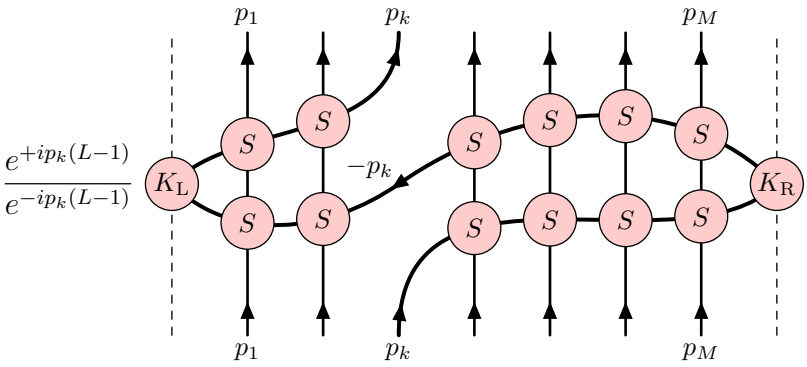
with boundary scattering factor

$$K_{\mathrm{R}}(+p) = e^{+ip}. \tag{3.66}$$

Compatibility of both boundaries leads to a set of Bethe equations for the open chain

$$\frac{\exp\big(i(L-1)(+p_k)\big)}{\exp\big(i(L-1)(-p_k)\big)} \frac{K_{\mathrm{R}}(+p_k)}{K_{\mathrm{L}}(-p_k)} \prod_{\substack{j=1 \\ j\neq k}}^{M} \frac{S(+p_k, p_j)}{S(-p_k, p_j)} = 1, \tag{3.67}$$

or in a graphical representation:



(3.68)

[19] The factors of $e^{\pm ip}$ were inserted to compensate for the plane wave factor at site $k = 1$.

[20] Up to normalisation, the exact eigenstates are invariant under flipping the sign of the momentum p because they are a superposition of ingoing and outgoing waves.

Note that these equations are invariant under flipping the sign of any momentum $p_j \to -p_j$. Flipping the sign of p_k inverts the equation.

The Bethe equations in rational form read

$$\left(\frac{u_k+\frac{i}{2}}{u_k-\frac{i}{2}}\right)^{2L} = \prod_{\substack{j=1\\ j\neq k}}^{M} \frac{u_k-u_j+i}{u_k-u_j-i}\,\frac{u_k+u_j+i}{u_k+u_j-i}\,. \tag{3.69}$$

Modified boundaries lead to some additional factors in the equations.

Bethe Equations for the XXZ Model. The XXX model is part a larger XXZ family of integrable models which are solvable by the above Bethe ansatz.[21] Strictly speaking the XXZ model is the model defined above. However, we can add a few parameters while preserving the features of the original model[22]

$$\begin{aligned}\mathcal{H}_{k,k+1} = {}& \alpha_1(1\otimes 1)+\alpha_2(\sigma^z\otimes 1)+\alpha_3(1\otimes\sigma^z)+\alpha_2(\sigma^z\otimes\sigma^z)\\ &+\alpha_5(\sigma^x\otimes\sigma^x+\sigma^y\otimes\sigma^y)+i\alpha_6(\sigma^x\otimes\sigma^y+\sigma^y\otimes\sigma^x).\end{aligned} \tag{3.70}$$

The 6 free parameter have the following meaning:

- one overall shift of energies: $\delta\alpha_1$,
- one trivial deformation for closed chains: $\delta\alpha_2 = -\delta\alpha_3$,
- one shift proportional to $\mathcal{J}^z$: $\delta\alpha_2 = +\delta\alpha_3$,
- one overall scaling of energies: $\delta\alpha_k = \alpha_k\delta\beta$,
- one quantum deformation parameter $\hbar$ also known as $q = e^{i\hbar}$ and the anisotropy $\Delta = \frac{1}{2}(q+q^{-1})$,
- one magnetic flux parameter ρ.

The resulting Bethe equations for closed chains read

$$\left(\frac{\sin\hbar(u_k+\frac{i}{2})}{\sin\hbar(u_k-\frac{i}{2})}\right)^{L} e^{i\rho L} = \prod_{\substack{j=1\\ j\neq k}}^{M} \frac{\sin\hbar(u_k-u_j+i)}{\sin\hbar(u_k-u_j-i)}\,. \tag{3.71}$$

These Bethe equations are called *trigonometric* as opposed to the *rational* Bethe equations for the XXX model.[23] The total momentum and energy are is given by

$$P = \sum_{k=1}^{M} p(u_k), \qquad E = \gamma_1 L + \gamma_2 M + \gamma_3 \sum_{k=1}^{M} e(u_k) \tag{3.72}$$

[21] The latter is part of the even larger XYZ family, but its solution requires more advances techniques because there is no U(1) symmetry to preserve the number of magnons.

[22] This is in fact the most general nearest neighbour Hamiltonian which commutes with $\mathcal{J}^z = \sum_k \sigma_k^z$.

[23] Both sets of Bethe equations can be written in either rational or trigonometric form with a suitable choice of variables, e.g. $z_k = \exp(i\hbar u_k)$ for XXZ. The distinguished set of variables, however, is where u_j appears only in the combination $u_j - u_k$. Using these variables the Bethe equations are rational and trigonometric for XXX and XXZ, respectively.

with

$$e^{ip(u)} = \frac{\sin\hbar(u+\frac{i}{2})}{\sin\hbar(u-\frac{i}{2})}\,, \qquad e(u) = p'(u). \tag{3.73}$$

Evidently, these equations reduce to the rational case in the limit $\hbar \to 0$.

S14]35

XXX model with Higher Spin. We can also use a different Hilbert space for the spin chain, for example a spin $s = 1$ representation spanned by three states $|0\rangle$, $|1\rangle$ and $|2\rangle$ corresponding to spin up, spin zero and spin down. The so-called XXX_1 Hamiltonian has SU(2) symmetry, in particular it preserves $\mathcal{J}^z$. Therefore, the Hamiltonian kernel takes a block-diagonal form in the basis E

$$\mathcal{H}_{k,k+1} = \left(\begin{array}{c|cc|ccc|cc|c} * & & & & & & & & \\ \hline & * & * & & & & & & \\ & * & * & & & & & & \\ \hline & & & * & * & * & & & \\ & & & * & * & * & & & \\ & & & * & * & * & & & \\ \hline & & & & & & * & * & \\ & & & & & & * & * & \\ \hline & & & & & & & & * \end{array}\right), \quad E = \left(\begin{array}{c} |00\rangle \\ \hline |10\rangle \\ |01\rangle \\ \hline |20\rangle \\ |11\rangle \\ |02\rangle \\ \hline |21\rangle \\ |12\rangle \\ \hline |22\rangle \end{array}\right). \tag{3.74}$$

We do not reproduce the coefficients because they do not add a qualitative insight.

The above Bethe ansatz works with small alterations:

- vacuum:

$$|0\rangle = |0\dots0\rangle. \tag{3.75}$$

- one-magnon states:

$$|p\rangle = \sum\nolimits_k e^{ipk} |\dots \overset{k}{1} \dots\rangle. \tag{3.76}$$

- two-magnon states:

$$\begin{aligned} |p<q\rangle &= \sum\nolimits_k e^{ipk+iql} |\dots \overset{k}{1} \dots \overset{l}{1} \dots\rangle, \\ |p;2\rangle &= \sum\nolimits_k e^{ipk} |\dots \overset{k}{2} \dots\rangle. \end{aligned} \tag{3.77}$$

The action of the Hamiltonian on partial eigenstates now yields some additional terms

$$\begin{aligned} (\mathcal{H}-E)|p<q\rangle &= \sum\nolimits_k e^{i(p+q)k} \big(*|\dots \overset{k}{1}1\dots\rangle + *|\dots \overset{k}{2}\dots\rangle\big), \\ (\mathcal{H}-E)|p;2\rangle &= \sum\nolimits_k e^{ipk} \big(*|\dots \overset{k}{1}1\dots\rangle + *|\dots \overset{k}{2}\dots\rangle\big). \end{aligned} \tag{3.78}$$

The scattering ansatz needs to be supplemented in order to compensate them appropriately.

$$|p,q\rangle = |p<q\rangle + S|q<p\rangle + C|p+q;2\rangle. \tag{3.79}$$

To construct the exact eigenstate we now have to solve two linear equations. The coefficient S is the scattering factor which is relevant for IR physics. The contact term C is important for the solution, but it merely describes the UV physics of the eigenstate.[24]

The resulting Bethe equations for a closed chain read

$$\left(\frac{u_k+i}{u_k-i}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{M} \frac{u_k-u_j+i}{u_k-u_j-i}\,, \qquad e^{ip} = \frac{u+i}{u-i}\,, \quad e(u) = p'(u). \tag{3.80}$$

Note that the Bethe equations are almost the same up to a different prefactor of i on the l.h.s. of the Bethe equations and likewise in the definition of the magnon momentum.

The generalisation to arbitrary spin s representations at each site is evident (and correct)

$$\left(\frac{u_k+is}{u_k-is}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{M} \frac{u_k-u_j+i}{u_k-u_j-i}\,, \qquad e^{ip} = \frac{u+is}{u-is}\,. \tag{3.81}$$

The corresponding model is called the XXX_s model.

Bethe Ansatz at Higher Rank. Generalisation of the XXX model to higher-rank groups exist. For example, consider a chain with $\mathrm{SU}(N)$ symmetry and spins in the fundamental representation

$$\mathbb{V} = \mathbb{C}^N, \qquad |1\rangle, \ldots, |N\rangle \in \mathbb{V}. \tag{3.82}$$

An integrable nearest neighbour Hamiltonian is given by the kernel

$$\mathcal{H}_{k,k+1} = \mathcal{I}_{k,k+1} - \mathcal{P}_{k,k+1}. \tag{3.83}$$

More explicitly, this kernel acts as follows

$$\mathcal{H}|ab\rangle = |ab\rangle - |ba\rangle. \tag{3.84}$$

We can again perform the Bethe ansatz:

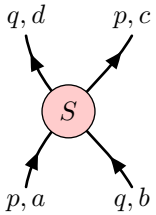
- vacuum:
$$|0\rangle = |1\ldots1\rangle. \tag{3.85}$$
- there are now $N-1$ flavours of one-magnon states labelled by $a = 2, \ldots N$
$$|p,a\rangle = \sum\nolimits_k e^{ipk}|\ldots \overset{k}{a}\ldots\rangle. \tag{3.86}$$

[24] The term $|p+2;2\rangle$ should be viewed as a contribution when both magnons reside on a single site. We did not have to consider such terms before because for spin 1/2 a single site can only be excited once.

- To accommodate for the various combinations of magnon flavours, we need a *scattering matrix*[25] instead of a scattering factor for the definition of two-magnon states

$$\big|(p,a),(q,b)\big\rangle = \big|(p,a)<(q,b)\big\rangle + \sum_{c,d=2}^{N} S_{ab}^{cd}(p,q)\,\big|(q,d)<(p,c)\big\rangle. \tag{3.87}$$

The S-matrix may again be represented graphically as follows:



(3.88)

The scattering matrix is a new feature for models based on a higher-rank algebra.

- The matrix can be computed as before by matching all asymptotic regions. In our case, one finds

$$S_{ab}^{cd}(u,v) = \frac{(u-v)\delta_a^c\delta_b^d + i\delta_a^d\delta_b^c}{u-v-i}\,. \tag{3.89}$$

- It preserves the residual SU$(N-1)$ of the magnons on the vacuum state.
- For $u\to\infty$ or $v\to\infty$ it is trivial

$$S_{ab}^{cd}(\infty,v) = S_{ab}^{cd}(u,\infty) = \delta_a^c\delta_b^d. \tag{3.90}$$

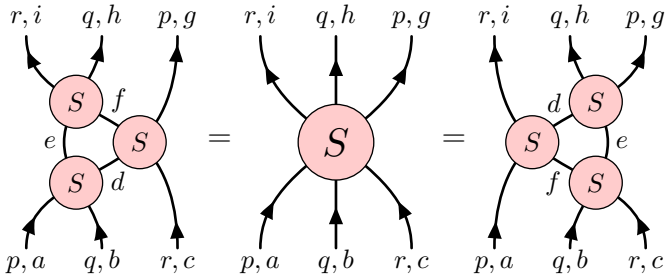
- For equal rapidities it reads

$$S_{ab}^{cd}(u,u) = -\delta_a^d\delta_b^c. \tag{3.91}$$

- It satisfies the *Yang–Baxter equation* which guarantees that states of factorised scattering can be defined consistently

$$S_{ab}^{de}(p,q)S_{dc}^{gf}(p,r)S_{ef}^{hi}(q,r) = S_{bc}^{ef}(q,r)S_{af}^{di}(p,r)S_{de}^{gh}(p,q). \tag{3.92}$$

The flow of indices is best explained using a figure:



(3.93)

An abbreviated version of the formal expression reads

$$S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12}. \tag{3.94}$$

This equation is a central relation for all quantum integrable systems.

[25] More precisely it is a tensor of rank 4, but when acting on two-magnon states it can be viewed as a matrix.

Nested Bethe Ansatz. The S-matrix now changes the flavour of the particles which are scattering. We thus cannot (easily) set up a consistency equation for periodic wave functions. We would like to "diagonalise" the S-matrix. However, there is no universal method to diagonalise a tensor, but this procedure has to be carefully designed for the problem in question:

- Step 1: Consider a new vacuum state

$$|0\rangle^2 = |2_1 2_2 \dots 2_M\rangle := \big|(p_1, 2), \dots, (p_M, 2)\big\rangle. \tag{3.95}$$

  The S-matrix is applied easily to this state because scattering is automatically a plain factor $S_{22}^{22}(p, q)$.
- Step 2: Introduce a new types of excitations on the above vacuum

$$\big|(u, a)\big\rangle^2 = \sum_{k=1}^{M} \psi_k(u) |2_1 \dots 2_{k-1} a_k 2_{k+1} \dots 2_M\rangle. \tag{3.96}$$

  There are now $N-2$ types of excitations labelled by $a = 3, \dots, N$. The new wave function $\psi_k(u)$ is not a plane wave because the vacuum state $|0\rangle^2$ is not homogeneous. It must be carefully chosen to enable an easy construction of scattering states and thus it depends on all the underlying magnon momenta p_k. We refrain from presenting the details.
- Step 3: Constructing states with two new excitations lead to a new S-matrix S^2 with $(N-2)^4$ components. This S-matrix has precisely the same form as the previous one but with fewer components.

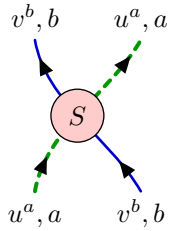
This procedure is reminiscent of the Bethe ansatz. In terms of states and excitations, we have achieved the following:

$$\begin{array}{ccccc} \text{spins} & & \text{magnons} & & \text{excitations} \\ |1\rangle & & |1\rangle & & |1\rangle \\ |2\rangle & \Longrightarrow & |1\rangle \to |2\rangle & \Longrightarrow & |1\rangle \to |2\rangle \\ |3\rangle & & |1\rangle \to |3\rangle & & |2\rangle \to |3\rangle \\ \vdots & & \vdots & & \vdots \\ |N\rangle & & |1\rangle \to |N\rangle & & |2\rangle \to |N\rangle \end{array} \tag{3.97}$$

The Bethe ansatz singles out the vacuum state $|1\rangle$ and converts all other spin states to magnon excitations $|1\rangle \to |a\rangle$ with $a = 2, \dots N$. The next step singles out one of the magnon excitations $|1\rangle \to |2\rangle$ and declares it as a new vacuum. The remaining magnons are obtained as new excitations $|2\rangle \to |a\rangle$ of the new vacuum with $a = 3, \dots N$. The procedure, called the *nested Bethe ansatz* can be iterated $N-1$ times in total. At the end we are left with

- the vacuum state $|1\rangle$,
- the magnon excitation $|1\rangle \to |2\rangle$,
- $N-2$ higher excitations $|a-1\rangle \to |a\rangle$ with $a = 3, \dots, N$.

Importantly, these interactions now all scatter diagonally, so the scattering matrix has been disintegrated into a collection of scattering factors $S^{a,b}(u^a, v^b)$



(3.98)

There is no mixing between the various flavours of excitations. It is in fact excluded by conservation of charges of the excitations.

For a given set of excitations, one can construct an eigenstate on the infinite chain. These arise as a sum over all admissible distributions of the excitations. In each distribution we must stack the excitations on the vacuum sites as towers without gaps

$$|4,3,1,4,1,1,2\rangle \longrightarrow \begin{array}{ccccccccc} 4(u_1^4) & & & & 4(u_2^4) & & & & \\ 3(u_1^3) & 3(u_2^3) & & & 3(u_3^3) & & & & \\ 2(u_1^2) & 2(u_2^2) & & & 2(u_3^2) & & & 2(u_4^2) & \\ |1\rangle & |1\rangle & |1\rangle & & |1\rangle & |1\rangle & |1\rangle & |1\rangle & \end{array} \tag{3.99}$$

The relative phase factors between two distributions are determined by hopping rules: There is a factor for moving one excitation on top of another excitation from the left.

$$\begin{array}{c} b(v^b) \\ \\ \\ \end{array} \xrightarrow{F^{a,b}(u^a,v^b)} \begin{array}{c} b(v^b) \\ a(u^a) \\ \\ \end{array} \tag{3.100}$$

This factor $F^{a,b}(u^a, v^b)$ depends on the flavours of the excitations and on their rapidities. Note that moving two excitations past each other yields their scattering factor

$$S^{a,b}(u^a, v^b) = \frac{F^{a,b}(u^a, v^b)}{F^{b,a}(v^b, u^a)}\,. \tag{3.101}$$

Bethe Equations for Higher Rank. For SU(N), the elements of the diagonalised scattering matrix simply read

$$\begin{aligned} S^{a,a}(u^a, v^a) &= \frac{u^a - v^a - i}{u^a - v^a + i}\,, \\ S^{a,a\pm1}(u^a, v^{a\pm1}) &= \frac{u^a - v^{a\pm1} + \frac{i}{2}}{u^a - v^{a\pm1} - \frac{i}{2}}\,, \\ S^{a,b}(u^a, v^b) &= 1 \quad \text{for } |a-b| > 1. \end{aligned} \tag{3.102}$$

It is straightforward to set up the Bethe equations for a closed chain. The Bethe equations for the level-1 magnons read

$$\left(\frac{u_k^1 + \frac{i}{2}}{u_k^1 - \frac{i}{2}}\right)^L = \prod_{\substack{j=1 \\ j\neq k}}^{M^1} \frac{u_k^1 - u_j^1 + i}{u_k^1 - u_j^1 - i} \prod_{j=1}^{M^2} \frac{u_k^1 - u_j^2 - \frac{i}{2}}{u_k^1 - u_j^2 + \frac{i}{2}}\,. \tag{3.103}$$

The Bethe equations for higher-level excitations take the form

$$1 = \prod_{j=1}^{M^{a-1}} \frac{u_k^a - u_j^{a-1} - \frac{i}{2}}{u_k^a - u_j^{a-1} + \frac{i}{2}} \prod_{\substack{j=1 \\ j\neq k}}^{M^a} \frac{u_k^a - u_j^a + i}{u_k^a - u_j^a - i} \prod_{j=1}^{M^{a+1}} \frac{u_k^a - u_j^{a+1} - \frac{i}{2}}{u_k^a - u_j^{a+1} + \frac{i}{2}}\,, \tag{3.104}$$

and the top-level equations read

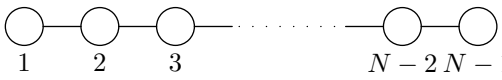
$$1 = \prod_{j=1}^{M^{N-2}} \frac{u_k^{N-1} - u_j^{N-2} - \frac{i}{2}}{u_k^{N-1} - u_j^{N-2} + \frac{i}{2}} \prod_{\substack{j=1 \\ j\neq k}}^{M^{N-1}} \frac{u_k^{N-1} - u_j^{N-1} + i}{u_k^{N-1} - u_j^{N-1} - i}\,. \tag{3.105}$$

The total momentum and energy are expressed as

$$e^{iP} = \prod_{k=1}^{M^1} \frac{u_k^1 + \frac{i}{2}}{u_k^1 - \frac{i}{2}}\,, \qquad E = \sum_{k=1}^{M^1} \left(\frac{i}{u_k^1 + \frac{i}{2}} - \frac{i}{u_k^1 - \frac{i}{2}} \right). \tag{3.106}$$

Inspecting these equations leads to the following generalisation to arbitrary simple Lie (super)algebras

- The interactions of the Bethe roots (r.h.s. of the Bethe equations) follow the symmetric Cartan matrix DA of the underlying group.[26] The latter is directly encoded into the Dynkin diagram of the corresponding algebra. For example, for $\mathrm{A}_{N-1} \simeq \mathrm{SU}(N)$:



(3.107)

- The momentum, energy and propagation of the Bethe roots (l.h.s. of the Bethe equations) follow the Dynkin labels of the underlying spin representation.

These equations have all of the generalisations discussed above: trigonometric deformations, introduction of magnetic fluxes, open chains, higher representations. One can also work make the spin chain inhomogeneous while preserving integrability. This can be achieved by a non-homogeneous (and typically non-local) Hamiltonian and by using site-dependent spin representations.

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[26]For the simply-laced groups A, D, E the Cartan matrix A is symmetric and $D = 1$. For the other groups $D = \mathrm{diag}(\ldots)$ makes the asymmetric Cartan matrix A symmetric in the product DA.

4 Quantum Integrability

Next we would like to discuss a formalism to deal with a large class of quantum integrable systems.

4.1 R-Matrix Formalism

In the above nested Bethe ansatz we have

- started with an SU(N) fundamental spin chain,
- reduced it to magnons with SU($N-1$) residual symmetry,
- reduced it further to excitations with SU($N-2$) residual symmetry,
- ...
- obtained a final set of excitations with no residual symmetry.

An interesting observation is that in every step we obtained a scattering matrix of the same form

$$S_{ab}^{cd}(u,v) = \frac{(u-v)\delta_a^c\delta_b^d + i\delta_a^d\delta_b^c}{u-v-i}\,. \tag{4.1}$$

In the first step the indices $a,b,c,d = 2,\dots,N$ took $N-1$ values, and in the last step they were all fixed $a,b,c,d = N$.

Since nothing much changed in each step of the nested Bethe ansatz, we can take a step backwards from the first level and consider the so-called *R-matrix*

$$R_{ab}^{cd}(u,v) = \frac{(u-v)\delta_a^c\delta_b^d + i\delta_a^d\delta_b^c}{u-v+i}\,, \tag{4.2}$$

where $a,b,c,d = 1,\dots N$. This matrix enjoys the full SU(N) symmetry of the spin chain as well as a couple of features to be discussed below which make it ideally suited for the construction and investigation of quantum integrable models. It differs from the above scattering matrices by an overall prefactor of $(u-v-i)/(u-v+i)$ which will be largely inconsequential but convenient.

R-Matrix Notations. Before we discuss the properties of R-matrices which come to use in the construction of integrable systems we shall introduce some notation for combining R-matrices which is very helpful for working out identities.

The R-matrix is a linear operator acting on the tensor square of the vector space $\mathbb{V} = \mathbb{C}^N$

$$\mathcal{R} : \mathbb{V}\otimes\mathbb{V} \to \mathbb{V}\otimes\mathbb{V}. \tag{4.3}$$

Moreover, it depends on one complex parameter associated to each of the two vector spaces, $\mathcal{R} = \mathcal{R}(u,v)$. Here it makes sense to also allow the point at infinity $u,v = \infty$ as a parameter value. In fact, the R-matrix depends only on the

difference of these parameters $\mathcal{R}(u,v) = \mathcal{R}(u-v)$, but we will hardly make use of this feature.

More concretely, the R-matrix takes the form

$$\mathcal{R}(u,v) = \frac{(u-v)\mathcal{I} + i\mathcal{P}}{u-v+i}\,, \tag{4.4}$$

where $\mathcal{I}$ and $\mathcal{P}$ denote the identity and permutation operators acting on $\mathbb{V}\otimes\mathbb{V}$. Note that the vector space $\mathbb{V}$ is a representation space of the Lie group $\mathrm{SU}(N)$, and the R-matrix is symmetric under the canonical action of $\mathrm{SU}(N)$ on the tensor product $\mathbb{V}\otimes\mathbb{V}$.

Introducing a basis $\{E^a\}$ for the vector space $\mathbb{V}$ and a dual basis $\{E_a\}$ for $\mathbb{V}^*$ we can decompose the R-matrix into components $\mathcal{R}^{cd}_{ab}$,

$$\mathcal{R} = (E^a \otimes E^b)\, R^{cd}_{ab}\, (E_c \otimes E_d). \tag{4.5}$$

In that sense, the R-matrix is actually a tensor of rank 4 with N^4 components (most of which are zero).[1] The components read

$$R^{cd}_{ab}(u,v) = \frac{(u-v)\delta^c_a\delta^d_b + i\delta^d_a\delta^c_b}{u-v+i}\,. \tag{4.6}$$

They are formulated in terms of Kronecker symbols δ^a_b which are invariant under $\mathrm{SU}(N)$ by construction

Let us now introduce an abbreviated symbolic and a graphical notation to deal with operators acting on tensor products of vector spaces $\mathbb{V}$ such as the R-matrix. In order to distinguish the vector spaces within the tensor product, each space receives a label $\mathbb{V}_k$.

In the symbolic notation, some operator $\mathcal{X}_{k,\dots,m}$ acts linearly on a tensor product of spaces

$$\mathcal{X}_{k,\dots,m} : \mathbb{V}_k \otimes \dots \otimes \mathbb{V}_m \to \mathbb{V}_k \otimes \dots \otimes \mathbb{V}_m. \tag{4.7}$$

This operator can also act on a tensor product with additional vector spaces, in which case it is assumed to act on the latter as the identity. For example, $\mathcal{X}_{23} := \mathcal{I}_1 \otimes \mathcal{X}_{23}$ when acting on $\mathbb{V}_1\otimes\mathbb{V}_2\otimes\mathbb{V}_3$ and $\mathcal{X}_{23} := \mathcal{I}_1 \otimes \mathcal{X}_{23} \otimes \mathcal{I}_4$ when acting on $\mathbb{V}_1\otimes\mathbb{V}_2\otimes\mathbb{V}_3\otimes\mathbb{V}_4$.

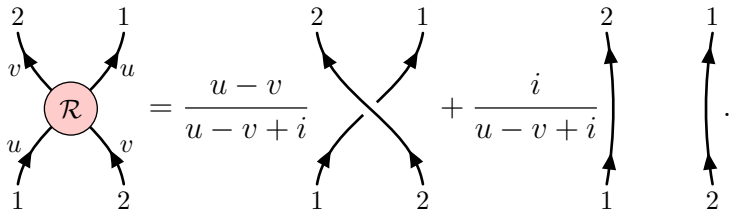
The R-matrix acts on a pair of spaces $\mathbb{V}_k$, $\mathbb{V}_l$ with associated parameters u_k, u_l. A useful shorthand notation is

$$\mathcal{R}_{k,l} := \mathcal{R}_{k,l}(u_k,u_l) : \mathbb{V}_k \otimes \mathbb{V}_l \to \mathbb{V}_k \otimes \mathbb{V}_l. \tag{4.8}$$

The short notation is sufficient because the parameters are linked tightly to the spaces.

[1] The term "matrix" refers to the fact that the R-matrix is a linear operator and can thus be written as a matrix. Often, R-matrices are written in $N^2 \times N^2$ matrix notation where the two ingoing and two outgoing indices are (implicitly) combined into a composite index.

In the graphical notation an operator is represented by some blob which has one ingoing and one outgoing leg for each vector space it acts upon. The R-matrix therefore has the following graphical representation:



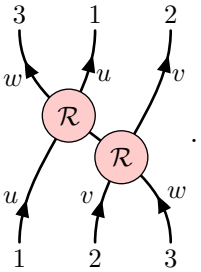
(4.9)

Each vector space has an associated parameter which is displayed next to the corresponding legs. The lines without operator blobs represent Kronecker symbols δ^a_b which are combined into identity or permutation operators, respectively.

Note that the R-matrix flips the ordering of the two legs in the graphical notation, whereas in the symbolic notation the ordering of the constituent vector spaces remains formally unchanged. It makes sense to consider the R-matrix as an operator which encodes the permutation of two vector spaces. Therefore, within a tangle of lines, one would expect to find an R-matrix at every intersection of two lines.

The above notations allow to conveniently combine operators acting on tensor products of vector spaces. For instance we can write or draw[2]

$\mathcal{R}_{13}\mathcal{R}_{23} =$



(4.10)

In components these expressions represent the combination

$$R^{df}_{ag}(u,w)R^{eg}_{bc}(v,w). \tag{4.11}$$

Note that the parameter w associated to $\mathbb{V}_3$ becomes an argument to both involved R-matrices.

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Properties of R-Matrices. The defining property of R-matrices is the Yang–Baxter equation

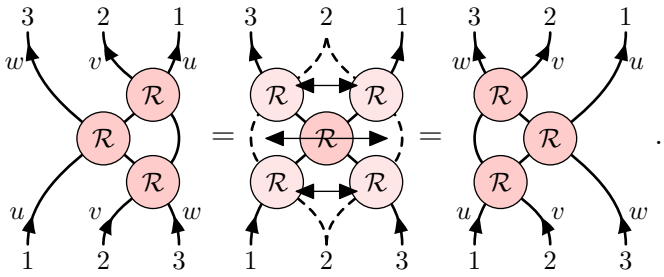
$$\mathcal{R}_{12}(u,v)\mathcal{R}_{13}(u,w)\mathcal{R}_{23}(v,w) = \mathcal{R}_{23}(v,w)\mathcal{R}_{13}(u,w)\mathcal{R}_{12}(u,v). \tag{4.12}$$

[2]There is some ambiguity in associating the flow of arrows to the order of multiplication of operators, and whether the latter is naturally from right to left or left to right. At the end of the day both choices are equivalent, but one has to stick to one convention. We shall assume the operators to be ordered from right to left along the flow of arrows.

In the context of the scattering matrix, this property is a prerequisite for factorised scattering. More concisely, the YBE can be written as

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{4.13}$$

The graphical representation of the YBE takes an inspiring form



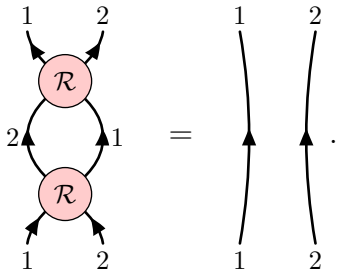
(4.14)

When considering a tangle of lines with the appropriate R-matrices at each intersection, the YBE allows us to shift one strand past an intersection of two other strands. As the figures shows the order of the R-matrices is inverted by such an operation.

Another important property is that $\mathcal{R}_{21}$ is the inverse of $\mathcal{R}_{12}$

$$\mathcal{R}_{21}\mathcal{R}_{12} = \mathcal{I}. \tag{4.15}$$

The graphical representation for the above property tells us that we can remove a double crossing of two strands and pull them straight



(4.16)

Note that for $\mathcal{R}_{12} = \mathcal{R}_{12}(u, v)$ the operator $\mathcal{R}_{21}$ is defined as

$$\mathcal{R}_{21} := \mathcal{R}_{21}(v, u) = \mathcal{P}_{12}\mathcal{R}_{12}(v, u)\mathcal{P}_{12} = \frac{(u-v)\mathcal{I} - i\mathcal{P}}{u - v - i}\,. \tag{4.17}$$

The combination of the above two rules leads to an interesting structure

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}, \qquad \mathcal{R}_{12}\mathcal{R}_{21} = \mathcal{I}\,. \tag{4.18}$$

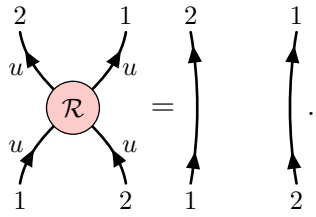
These are in fact the defining relations of the permutation group, where $\mathcal{R}_{k,l}$ represents a pairwise permutation between two elements labelled k and l. This feature allows to use R-matrices as the pairwise scattering matrix for a factorised scattering problem because for every permutation there is a unique combination of pairwise R-matrices up to identities. For a tangle of lines, the above identities imply that only the permutation between the ingoing and outgoing vector spaces

matters. In other words, lines can be deformed at will as long as at every crossing an R-matrix is inserted.

In addition, there are two properties related to special points which will be useful for the construction of integrable systems.

When both parameters are the same, $u = v$, the R-matrix becomes the permutation

$$\mathcal{R}(u,u) = \mathcal{P},$$



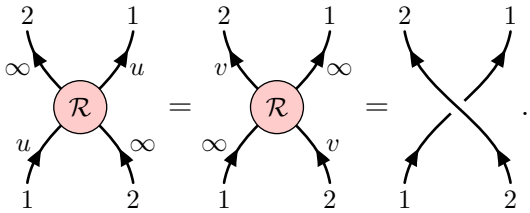
(4.19)

In the scattering context, this identity implies identical particles.

For the class of rational R-matrices the points $u = \infty$ or $v = \infty$ are also special. Here the R-matrix trivialises to the identity

$$\mathcal{R}(u,\infty) = \mathcal{R}(\infty,v) = \mathcal{I}, \tag{4.20}$$

or graphically



(4.21)

In the scattering context, this identity relates a magnon at zero momentum to a symmetry of the system.

S14]30

R-Matrix Generalisations. Note that the above R-matrix is one of the simplest ones that exist; there are many much more elaborate generalisations. Let us summarise a few of them briefly which share most of the above properties:

- The space $\mathbb{V}$ can be replaced by different representation spaces potentially of a different Lie group.
- The above R-matrix has no parameters beyond those associated to the two vector spaces. Most R-matrices allows for several globally defined deformation parameters. These deformation parameters may alter or spoil the properties associated to the special point $u, v = \infty$.
- Our R-matrix was defined on the tensor square of the space $\mathbb{V}$. R-matrices can also be defined for tensor products of inequivalent spaces $\mathbb{V}_k$, $\mathbb{V}_k$. In this case many of the discussed properties only hold when introducing one R-matrix for each pair of admissible spaces. Alternatively, one could consider the direct sum of all admissible vector spaces with a single R-matrix acting on the tensor square.

- We considered the case where every vector space has one associated parameter. Generalisations of this concept involve several (or no) parameters associated to a vector space.
- The R-matrix considered above depends on the difference of its two parameters. Most known examples obey such a difference form. There are, however, notable exceptions where the R-matrix is not of difference form (even after taking into account reparametrisations discussed below).
- One can apply a map $u \mapsto f(u)$ to the parameters of $\mathcal{R}$ without spoiling most of the properties discussed above (the location of the special points evidently changes). For example, our S-matrix was initially given in terms of momenta p, q instead of rapidities u, v. The difference property, however, singles out a preferred choice of parameters.
- For most purposes, the overall normalisation of the R-matrix does not matter. For instance, one often considers polynomial R-matrices without the denominator of our R-matrix was removed. Such extra factors modify some of the above relations slightly. For instance, $\mathcal{R}_{21}\mathcal{R}_{12}$ will equal the identity merely up to some overall factor.

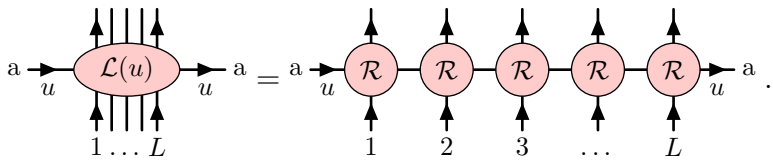
S14]45

Monodromy and Transfer Matrices. In classical field theory, we introduced the Lax connection as a formulation of integrability. For spin chains, however, space is discrete and states are quantum. Therefore, a Lax connection is not applicable; it is replaced by the R-matrix. The R-matrix takes the analogous role of the parallel transport of the Lax connection past one site of the spin chain which is the minimum distance that makes sense for this model.[3] Moreover, the R-matrix is a quantum operator rather than a function of phase space.

Next let us now consider the spin chain as a whole. We assume the chain to have closed boundary conditions. Using the above analogy with classical field theory, the monodromy matrix $\mathcal{L}(u)$ for the spin chain (which describes half of the Lax pair) is pieced together from an R-matrix for each site

$$\mathcal{L}_{\mathrm{a}}(u) = \mathcal{R}_{\mathrm{a},L}\mathcal{R}_{\mathrm{a},L-1}\ldots\mathcal{R}_{\mathrm{a},2}\mathcal{R}_{\mathrm{a},1}. \tag{4.22}$$

In graphical notation the monodromy matrix reads



(4.23)

The monodromy matrix $\mathcal{L}_{\mathrm{a}}(u)$ is a matrix of operators which act on the Hilbert space of the quantum spin chain. Note that the parameter $u = u_{\mathrm{a}}$ is associated to the auxiliary space $\mathbb{V}_{\mathrm{a}}$ on which the matrix acts (i.e. the space of the classical Lax

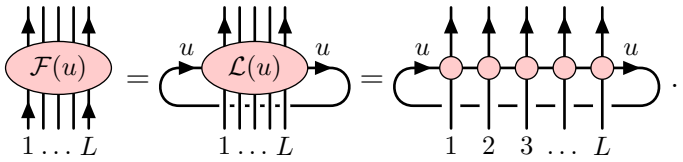
[3]The R-matrix serves many purposes in spin chains. In this context it is also known as the Lax matrix, typically with a convenient prefactor and one of the two parameters u, v fixed to a particular value.

pair). The parameters v_k associated to the spin sites are fixed to some values. Since we are interested in *homogeneous* spin chain models we choose all parameters to be equal, conveniently $v_k = 0$.[4]

It is now straight-forward to construct charges in involution as the trace of the monodromy matrix

$$\mathcal{F}(u) = \mathrm{tr}_{\mathrm{a}}\, \mathcal{L}_{\mathrm{a}}(u). \tag{4.24}$$

This so-called transfer matrix $\mathcal{F}(u)$ can be written graphically as[5]

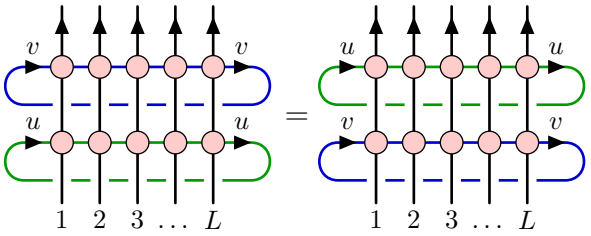


(4.25)

Two transfer matrices commute at arbitrary spectral parameters u, v

$$\big[\mathcal{F}(u), \mathcal{F}(v)\big] = 0. \tag{4.26}$$

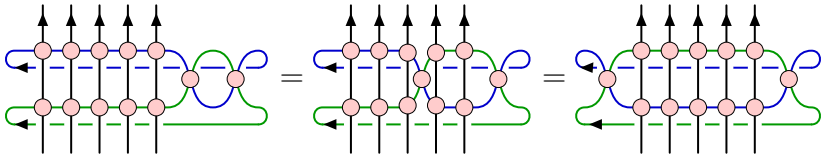
To see this is quite evident in graphical notation where we need to show the following equality



(4.27)

We have already learned that we can deform the lines and move them past others lines and intersections. This allows to move the upper loop past the lower loop and thus switch their ordering.

A symbolic proof is also straight-forward, but requires several steps which are not as easy to spot in a long sequences of symbols. The first step is to let the two loops overlap somewhere by inserting an R-matrix and its inverse. The next step consists in pulling the upper loop below the lower loop past all intersections along the spin chain. In a final step the R-matrix and its inverse are removed by pulling the loops apart. These three steps look as follows



(4.28)

[4] An *inhomogeneous* spin chain with $v_k \neq v_l$ is integrable as well and can be treated with minor modifications.

[5] Since the chain is periodic, the lines should be drawn on the surface of a cylinder. The loop can thus be closed without introducing further crossings.

Local Charges. Next we have to define a suitable Hamiltonian for the system. This should be constructed out of the transfer matrix $\mathcal{F}(u)$ such that it automatically commutes with the transfer matrices $\mathcal{F}(v)$ at arbitrary values v. We shall see that the expansion around the point $u = 0$ (which coincides with the values of the parameters v_k associated to the spin sites) is perfectly suited for this purpose.

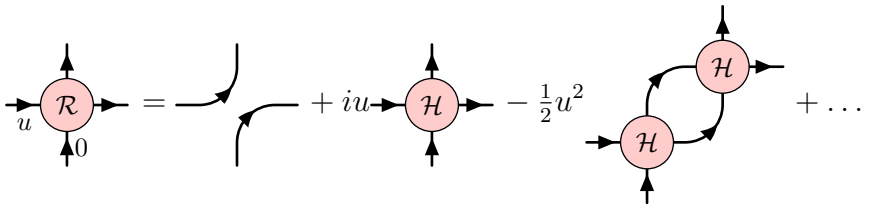
Let us therefore expand the R-matrix around the point $u = 0$ with fixed $u_k = 0$. We find

$$\mathcal{R}_{\mathrm{a},k}(u,0) = \mathcal{P}_{\mathrm{a},k} + iu\mathcal{P}_{\mathrm{a},k}\mathcal{H}_{\mathrm{a},k} - \tfrac{1}{2}u^2\mathcal{P}_{\mathrm{a},k}\mathcal{H}^2_{\mathrm{a},k} + \dots \tag{4.29}$$

with the Hamiltonian kernel of the Heisenberg XXX model or its generalisation to the SU(N) fundamental spin chain

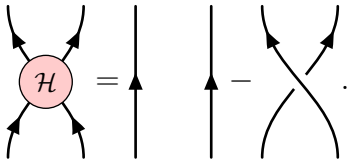
$$\mathcal{H}_{k,l} = \mathcal{I}_{k,l} - \mathcal{P}_{k,l}. \tag{4.30}$$

In a graphical notation the expansion can be written as



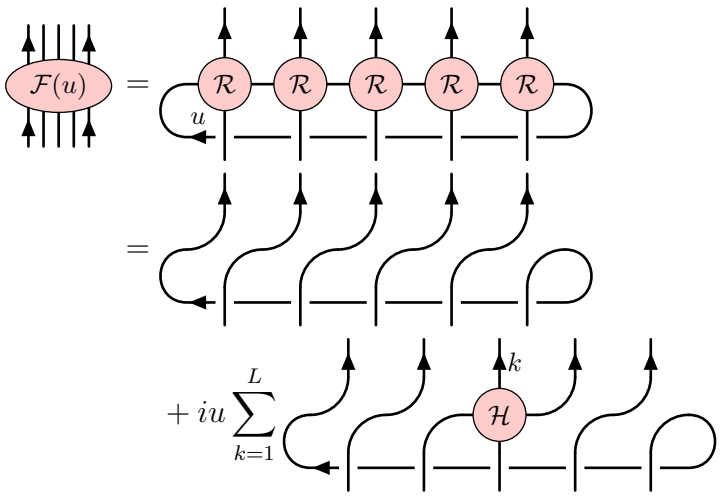
(4.31)

with the Hamiltonian kernel taking the form[6]

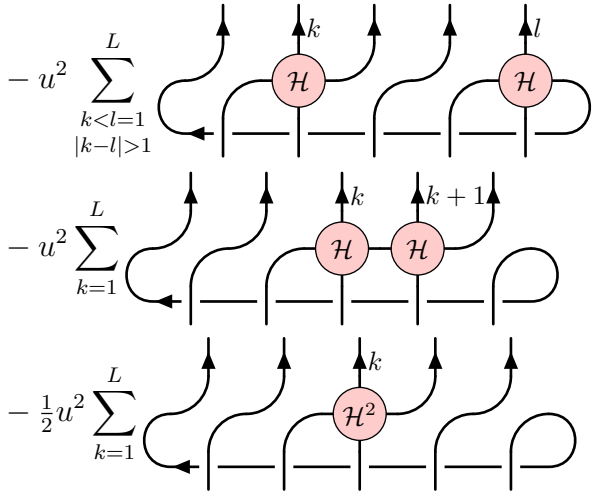


(4.32)

Now we can expand the transfer matrix $\mathcal{F}(u)$ around the point $u = 0$. Up to second order we find



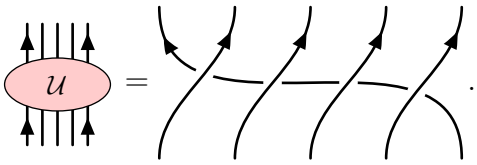
[6]We use a different convention to for the R-matrix and kernels of local operators: The R-matrix is located at the intersection of two crossing lines, whereas the Hamiltonian joins two lines which do not cross. This implies a different ordering for the external legs.



(4.33)

Let us discuss the arising terms one at a time:

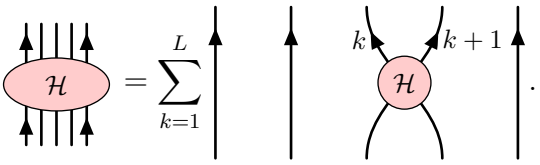
The leading term describes a cyclic shift $\mathcal{U}$ of the closed chain



(4.34)

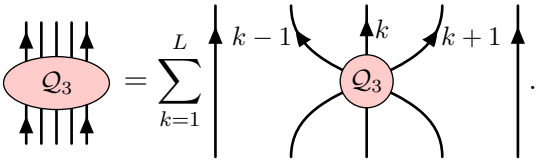
Also the sub-leading terms are cyclic shift operations for most of the legs, so it makes sense to factor out the operator $\mathcal{U}$.

We denote the term at linear order in u by $-iu\mathcal{U}\mathcal{H}$. The operator $\mathcal{H}$ is in fact the Hamiltonian given by a homogeneous sum of Hamiltonian kernels around the closed chain



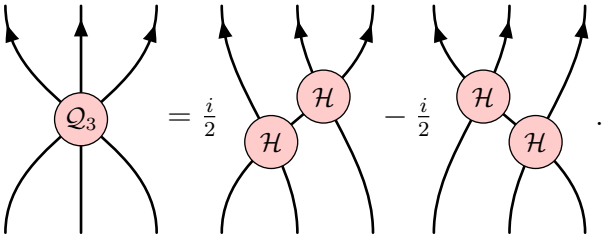
(4.35)

The term on the next line contains two insertions of the Hamiltonian kernels at arbitrary non-overlapping positions of the spin chain. All of these terms are generated by the square of the Hamiltonian $-\frac{1}{2}u^2\mathcal{U}\mathcal{H}^2$. However, one has to pay attention to the terms where two kernels overlap: The terms where two kernels are inserted at the same location is covered precisely by the last line. The second but last line describes terms where the insertions are shifted by one site. Those terms arise in $\mathcal{H}^2$, but only with half of the coefficient. Conversely, there are further terms in $\mathcal{H}^2$ where the order of insertions is flipped. We summarise these additional and missing terms in the operator



(4.36)

It is a local operator with kernel that acts on next-to-nearest neighbours



(4.37)

In formulas we can write this kernel as

$$\mathcal{Q}_{3;k,l,m} = \tfrac{i}{2}[\mathcal{H}_{l,m}, \mathcal{H}_{k,l}]. \tag{4.38}$$

Altogether we find that the expansion is written nicely as an exponential

$$\mathcal{F}(u) = \mathcal{U} \exp\bigl(iu\mathcal{H} + iu^2\mathcal{Q}_3 + \ldots\bigr). \tag{4.39}$$

The operators $\mathcal{Q}_r$ in the exponent have the relevant property of being *local*. In fact, their kernels extend over an extended range of r sites. In particular, the first operator $\mathcal{Q}_2$ in this tower is the Hamiltonian $\mathcal{Q}_2 = \mathcal{H}$. Commutativity of the transfer matrices $\mathcal{F}(u)$ and $\mathcal{F}(v)$ at arbitrary values u, v leads to the involution property

$$[\mathcal{Q}_r, \mathcal{Q}_s] = 0. \tag{4.40}$$

We have thus constructed a tower of commuting local operators.

Locality of the integrable charges is a crucial feature for at least two reasons:

- For a Hilbert space of dimension N there always exist $N-1$ commuting independent operators which also commute with a given Hamiltonian: In a basis where the Hamiltonian is diagonal, these are the remaining independent diagonal matrices. Since this construction does not rely on any special properties of the physical system, it can hardly be useful. In order to be useful, quantum integrability must require further properties for the commuting charges such as locality.
- In the magnon scattering picture, local charges act on the magnons individually provided that the latter are sufficiently well separated. For a state of m magnons, m local commuting charges are needed to guarantee that the momenta are individually conserved. Since the local charges do not distinguish the ordering of magnons along the chain, there are $m!$ partial eigenstates with degenerate charge eigenvalues. These are related by the factorised scattering matrix.

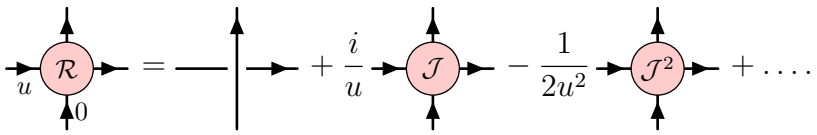
Multi-Local Charges. Another point of interest is $u = \infty$. Here the R-matrix has the following expansion

$$\mathcal{R}_{\mathrm{a},k}(u,0) = \mathcal{I}_{\mathrm{a},k} + iu^{-1}\mathcal{J}_{\mathrm{a},k} - \tfrac{1}{2}u^{-2}\mathcal{J}^2_{\mathrm{a},k} + \ldots\,, \tag{4.41}$$

with the operator

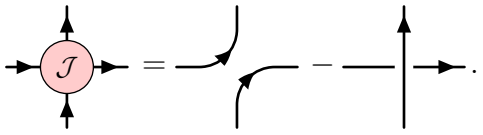
$$\mathcal{J}_{\mathrm{a},k} = \mathcal{P}_{\mathrm{a},k} - \mathcal{I}_{\mathrm{a},k}. \tag{4.42}$$

The expansion in terms of figures reads



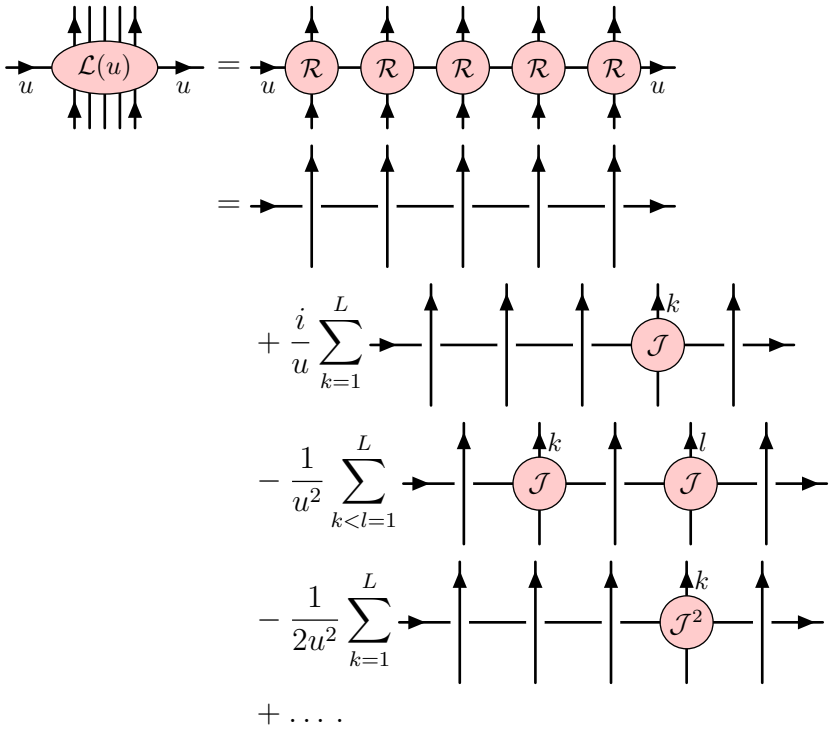
(4.43)

and the operator $\mathcal{J}$ takes the form[7]



(4.44)

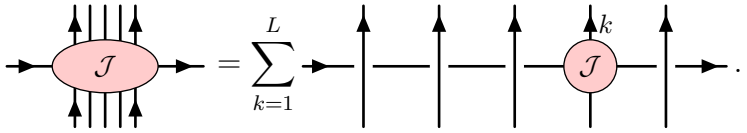
We now expand the monodromy matrix $\mathcal{L}_{\rm a}(u)$ to second order



(4.45)

Let us again discuss the terms that arise: The leading term is merely the identity operator.

At first order we find an operator $\mathcal{J}_{\rm a}$ which is the sum over the insertions of $\mathcal{J}_{{\rm a},k}$ at every site k of the spin chain

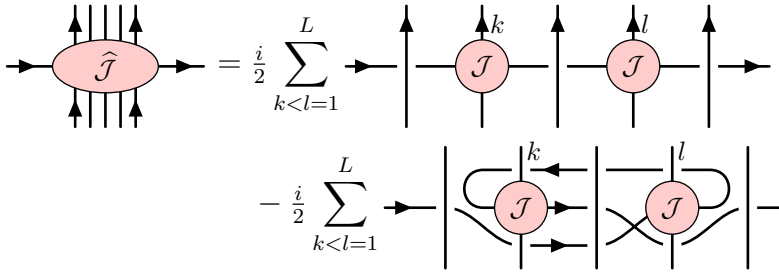


(4.46)

It turns out that $\mathcal{J}_{{\rm a},k}$ can be viewed as a fundamental representation of U(N) acting on site k of the chain. Consequently, $\mathcal{J}_{\rm a}$ is the tensor product representation of U(N) on the whole chain.

[7]Here we use the same convention as with the R-matrix that $\mathcal{J}$ resides at the intersection of two lines.

At the next order we find again two insertions of $\mathcal{J}_{\mathrm{a},k}$. These terms arise from $-\frac{1}{2}u^{-2}\mathcal{J}_{\mathrm{a}}^2$ which lets $\mathcal{L}_{\mathrm{a}}$ take the form of an exponential. As before, the coefficients of the terms differ by a factor of 2, and an equal number of terms with a different ordering is missing. We summarise them in an operator $\widehat{\mathcal{J}}_{\mathrm{a}}$



(4.47)

This operator turns out to be the generator of an extended symmetry to be discussed further below.

Altogether we find for the expansion of the monodromy

$$\mathcal{L}_{\mathrm{a}}(u) = \exp\bigl(iu^{-1}\mathcal{J}_{\mathrm{a}} + iu^{-2}\widehat{\mathcal{J}}_{\mathrm{a}} + \ldots\bigr). \tag{4.48}$$

From the above discussion it should be evident that the expansion of $\mathcal{L}(u)$ around $u = \infty$ yields a tower of *multi-local* charges. These act at several sites of the spin chain at the same time. The form of the expansion is very special, and related to the fact that the R-matrix reduces to the identity matrix at the point $u = \infty$. At generic points u, the monodromy matrix expands into a set of operators which act non-locally on the spin chain without apparent order.

4.2 Other Types of Bethe Ansätze

The Bethe equations describe the spectrum of quantum spin chains, but there are several ways in which they can be derived and formulated. The various approaches lead to different perspectives, which may be particularly helpful in addressing specific kinds of problems. In the following we present the main few approaches.

Algebraic Bethe Ansatz. We can apply the R-matrix formalism to construct eigenstates of the closed spin chain. This method is not only closer to the quantum field theory formalism, but it also largely based in algebra.

Let us first investigate the monodromy matrix for the Heisenberg XXX spin chain with $N = 2$. The monodromy matrix $\mathcal{L}$ is a 2×2 matrix acting on the auxiliary spin site (as well as a big matrix acting on the space of the spin chain)

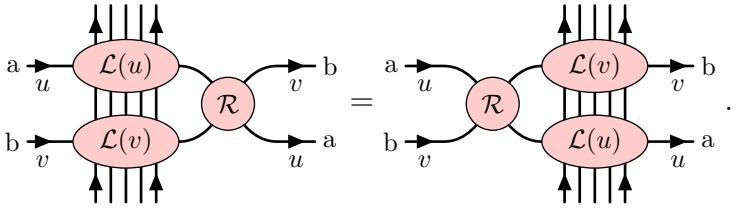
$$\mathcal{L}(u) = \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & \mathcal{D}(u) \end{pmatrix}. \tag{4.49}$$

The components $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are operators acting on the spin chain which obey certain commutation relations. These can be summarised in the so-called

RTT-relations[8]

$$\mathcal{R}_{\mathrm{ab}}(u,v)\mathcal{L}_{\mathrm{a}}(u)\mathcal{L}_{\mathrm{b}}(v) = \mathcal{L}_{\mathrm{a}}(v)\mathcal{L}_{\mathrm{b}}(u)\mathcal{R}_{\mathrm{ab}}(u,v). \tag{4.50}$$

In the graphical notation they read



(4.51)

They follow straight-forwardly from repeated application of the Yang–Baxter equations.[9]

We know that our R-matrix is invariant under SU(2), in particular, it preserves the number of up and down spins. Consequently, a spin flip in the auxiliary space must be compensated by an opposite spin flip on the spin chain. The components $\mathcal{A}$ and $\mathcal{D}$ do not change the number of up and down spins, whereas $\mathcal{B}$ and $\mathcal{C}$ increase and decrease the number of up spins by one unit, respectively.

Recalling that we treated a spin flip as a magnon particle, the above discussion is reminiscent of the framework of quantum field theory where $\mathcal{B}$ and $\mathcal{C}$ take the roles of creation and annihilation operators, respectively, whereas $\mathcal{A}$ and $\mathcal{D}$ serve as charges. The RTT relations provide the commutation relations which are of the same kind as the commutation relations for (free) particles, but somewhat more involved.

To construct eigenstates we start again with a ferromagnetic vacuum state

$$|0\rangle = |\downarrow\downarrow\ldots\downarrow\rangle. \tag{4.52}$$

This state is evidently annihilated by $\mathcal{C}(u)$ for any u. Excited states are generated by acting with several $\mathcal{B}(u)$'s on the vacuum.

$$|u_1,\ldots,u_M\rangle = \mathcal{B}(u_1)\ldots\mathcal{B}(u_M)|0\rangle. \tag{4.53}$$

This state has M up spins, and therefore it is an M-magnon state. The u_k correspond to the magnon rapidities which are related to the magnons by the relation $u_k = \frac{1}{2}\cot(\frac{1}{2}p_k)$ we used earlier to introduce the rapidity variables. The operator $\mathcal{B}(u_k)$ places a magnon with momentum p_k on top of the existing magnon. This is done precisely in accordance with the rules to assemble multi-magnon states described above. All of this construction is neatly encoded into the R-matrix.

So far we have not specified boundary conditions because the monodromy matrix simply end at the first and the last sites. For a closed chain the latter should be

[8] The name originates from a notation where the monodromy matrix is assigned the letter T.

[9] The RTT-relations imply that the monodromy matrix $\mathcal{L}(u)$ is an R-matrix as well. This R-matrix is slightly more general than the one we discussed above: It acts on two inequivalent spaces, the auxiliary spin site and the Hilbert space of the spin chain. The spin chain space supplies not just one parameter v, but rather one parameter v_k for each spin site. We merely *decided* to set all these parameters to zero $v_k = 0$.

related as any other pair of adjacent sites. This is achieved by the trace within the transfer matrix $\mathcal{F}(u)$. We are thus interested in its eigenvalues which includes the energy spectrum of the closed chain. Let us therefore act with

$$\mathcal{F}(u) = \mathcal{A}(u) + \mathcal{D}(u) \tag{4.54}$$

on a state $|u_1, \ldots, u_M\rangle$. This operation can be performed by means of the RTT algebra of the components $\mathcal{A}, \mathcal{B}, \mathcal{D}$. Not too surprisingly, the state is an eigenstate of $\mathcal{F}(u)$ precisely if the magnon rapidities u_k satisfy the closed chain Bethe equations

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L = \prod_{\substack{j=1 \\ j\neq k}}^{M} \frac{u_k - u_j + i}{u_k - u_j - i} \qquad \text{for } k = 1, \ldots, M. \tag{4.55}$$

Interestingly, we can now compute the eigenvalue $F(u)$ of $\mathcal{F}(u)$ with full dependence on u

$$F(u) = \prod_{k=1}^{M} \frac{u - u_k - \frac{i}{2}}{u - u_k + \frac{i}{2}} + \left(\frac{u}{u+i}\right)^L \prod_{k=1}^{M} \frac{u - u_k + \frac{3i}{2}}{u - u_k + \frac{i}{2}}\,. \tag{4.56}$$

The two terms roughly correspond to action of the operators $\mathcal{A}(u)$ and $\mathcal{D}(u)$ up to a bunch of extra contributions which cancel between the two terms when the Bethe equations hold.

Above, we have derived a relationship between the expansion of $\mathcal{F}(u)$ at $u = 0$ and some local charges including the Hamiltonian

$$\mathcal{F}(u) = \mathcal{U} \exp\bigl(iu\mathcal{H} + iu^2\mathcal{Q}_3 + \ldots\bigr). \tag{4.57}$$

The same relationship evidently holds for the eigenvalues. We thus find[10]

$$\begin{aligned} F(u) &= U \exp\bigl(iuE + iu^2 Q_3 + \ldots\bigr), \\ U &= \prod_{k=1}^{M} \frac{-u_k - \frac{i}{2}}{-u_k + \frac{i}{2}}\,, \\ E &= \sum_{k=1}^{M} \left(\frac{i}{u_k + \frac{i}{2}} - \frac{i}{u_k - \frac{i}{2}}\right), \\ Q_3 &= \sum_{k=1}^{M} \left(\frac{i}{2(u_k + \frac{i}{2})^2} - \frac{i}{2(u_k - \frac{i}{2})^2}\right). \end{aligned} \tag{4.58}$$

A benefit of this so-called *algebraic Bethe ansatz* is that it is readily generalised to bigger symmetry algebras. Let us sketch how to apply to apply the algebraic

[10] All of the *local* charge eigenvalues originate from the first term only because the second term is suppressed by u^L. At sufficiently large order this term also contributes, but the corresponding charges can hardly be called local because they extend over whole length of the spin chain.

Bethe ansatz to the SU(N) fundamental spin chain. We first decompose the monodromy matrix as follows

$$\mathcal{L} = \begin{pmatrix} \mathcal{A}^1 & \mathcal{B}^1 & * & * \\ \mathcal{C}^1 & \mathcal{A}^2 & \ddots & * \\ * & \ddots & \ddots & \mathcal{B}^{N-1} \\ * & * & \mathcal{C}^{N-1} & \mathcal{A}^N \end{pmatrix}. \tag{4.59}$$

The operators $\mathcal{B}^r$ and $\mathcal{C}^r$ serve as creation and annihilation operators of $N-1$ kinds. They correspond one-to-one to the various excitations of the nested Bethe ansatz. The operators $\mathcal{A}^r$ on the diagonal leaves the numbers of all kinds of excitations invariant.[11] We will not need the other operators explicitly because they can be written as combinations of the above elementary building blocks. A generic state is then written as

$$|u_k^r, u_l^s, \ldots\rangle = \mathcal{B}^r(u_k^r)\mathcal{B}^s(u_l^s)\ldots|0\rangle. \tag{4.60}$$

where the vacuum is again the state with all spins aligned such that it is annihilated by all $\mathcal{C}^r$.

Analytic Bethe Ansatz. Let us reconsider the eigenvalue of the transfer matrix from the algebraic Bethe ansatz

$$F(u) = \prod_{k=1}^{M} \frac{u-u_k-\frac{i}{2}}{u-u_k+\frac{i}{2}} + \left(\frac{u}{u+i}\right)^L \prod_{k=1}^{M} \frac{u-u_k+\frac{3i}{2}}{u-u_k+\frac{i}{2}}. \tag{4.61}$$

Compare this to the definition of the transfer matrix

$$\mathcal{F}(u) = \mathrm{tr}_{\mathrm{a}}\, \mathcal{R}_{\mathrm{a},L}\mathcal{R}_{\mathrm{a},L-1}\ldots\mathcal{R}_{\mathrm{a},2}\mathcal{R}_{\mathrm{a},1}, \quad \mathcal{R}_{\mathrm{a},k} = \frac{u\mathcal{I}_{\mathrm{a},k} + i\mathcal{P}_{\mathrm{a},k}}{u+i}. \tag{4.62}$$

One immediately observes that $\mathcal{F}(u)$ is a rational function with an L-fold pole at $u=-i$, no other poles and $\mathcal{F}(\infty) = 2\mathcal{I}$. The eigenvalue $F(u)$ has the same properties, but additional apparent poles at $u = u_k - \frac{i}{2}$. How do these observations fit together? Did something go wrong?

Let us therefore investigate the residue of $F(u)$ at the dynamical pole $u = u_k - \frac{i}{2}$

$$\begin{aligned} & F(u_k - \tfrac{i}{2} + \epsilon) \\ \sim\ & -\frac{i}{\epsilon}\prod_{\substack{j=1\\ j\neq k}}^{M} \frac{u_k-u_j+i}{u_k-u_j} + \frac{i}{\epsilon}\left(\frac{u_k-\frac{i}{2}}{u_k+\frac{i}{2}}\right)^L \prod_{\substack{j=1\\ j\neq k}}^{M} \frac{u_k-u_j-i}{u_k-u_j} \\ \sim\ & -\frac{i}{\epsilon}\prod_{\substack{j=1\\ j\neq k}}^{M} \frac{u_k-u_j+i}{u_k-u_j}\left(1 - \left(\frac{u_k-\frac{i}{2}}{u_k+\frac{i}{2}}\right)^L \prod_{\substack{j=1\\ j\neq k}}^{M} \frac{u_k-u_j-i}{u_k-u_j+i}\right). \end{aligned} \tag{4.63}$$

[11] In the context of Lie algebra the above operators correspond to *Chevalley–Serre generators* and *simple roots* of the algebra.

This shows that the residue is zero whenever the Bethe equations are satisfied, and there are no unwanted dynamical poles.

We can use the above observations to formulate the *analytic Bethe ansatz*: Suppose we are given the above form of the transfer matrix eigenvalue function $F(u)$ with unknown Bethe roots u_k. The u_k are then constrained by the requirement that $F(u)$ has no poles other than an L-fold pole at $u = -i$. In other words, $F(u)$ must be an analytic function except at $u = -i$.

Baxter Equation. Recall the eigenvalue of the transfer matrix and introduce the slightly modified but more symmetric function $T(u) = (u + \frac{i}{2})^L F(u - \frac{i}{2})$

$$T(u) = (u + \tfrac{i}{2})^L \prod_{k=1}^{M} \frac{u - u_k - i}{u - u_k} + (u - \tfrac{i}{2})^L \prod_{k=1}^{M} \frac{u - u_k + i}{u - u_k}\,. \tag{4.64}$$

This function is polynomial of degree L with leading term $2u^L$. Furthermore, introduce the so-called Baxter Q-function $Q(u) = \prod_{k=1}^{M}(u - u_k)$. The above expression takes the form

$$T(u)Q(u) = (u + \tfrac{i}{2})^L Q(u - i) + (u - \tfrac{i}{2})^L Q(u + i). \tag{4.65}$$

On the one hand, the equation defines $T(u)$ as a function of $Q(u)$. On the other hand, it takes the form of a difference equation for $Q(u)$ which is known as the *Baxter equation*.

An important insight is: With the further assumption that $T(u)$ and $Q(u)$ are unknown *polynomials*, the Baxter equation becomes equivalent to the Bethe equations! Some comments

- The roots of the polynomial $Q(u)$ are the Bethe roots.
- $T(u)$ describes the transfer matrix for a given set of Bethe roots encoded into $Q(u)$.
- For any given $T(u)$, there are two solutions of the Baxter equation because the difference equation is of second order.
- The difference equation can be viewed as a quantisation of a differential equation describing classical physics.
- The Baxter equation generalises to many other integrable systems. In particular it can formulated for models where the coordinate Bethe ansatz does not apply, such as the Heisenberg XYZ chain. In the latter example, the functions T and Q are not polynomials but rather elliptic functions with two periodicities on the complex plane.

T-System. We defined the transfer matrix as a trace of a monodromy matrix with an auxiliary space transforming in the fundamental representation of SU(N). The concept of transfer matrices can be generalised easily to auxiliary spaced transforming in higher representations. The higher transfer matrices all commute with each other at arbitrary parameters.

The eigenvalue of the spin-1 transfer matrix for the Heisenberg XXX spin chain reads

$$T_1(u) = (u+i)^L \prod_{k=1}^{M} \frac{u-u_k-\frac{3i}{2}}{u-u_k+\frac{i}{2}} + (u-i)^L \prod_{k=1}^{M} \frac{u-u_k+\frac{3i}{2}}{u-u_k-\frac{i}{2}} + u^L \prod_{k=1}^{M} \frac{u-u_k-\frac{3i}{2}}{u-u_k+\frac{i}{2}} \frac{u-u_k+\frac{3i}{2}}{u-u_k-\frac{i}{2}} \,. \tag{4.66}$$

We have written the eigenvalue as a polynomial in analogy to $T_{1/2}(u) := T(u)$ vs. the original rational function $F(u)$.

These transfer matrices do not necessarily carry additional information, they merely reshuffle the available information. For instance there is a simple relationship between $T_{1/2}$ and T_1

$$T_{1/2}(u+\tfrac{i}{2})T_{1/2}(u-\tfrac{i}{2}) = u^L T_1(u) + (u+i)^L (u-i)^L. \tag{4.67}$$

This identity neatly reflects the SU(2) multiplication rule $(\frac{1}{2}) \otimes (\frac{1}{2}) = (1) \oplus (0)$. The relationship can be understood as follows: We first act with two monodromy matrices with fundamental auxiliary sites on the state. To turn them into transfer matrices, we should take a trace on each auxiliary space (l.h.s.). However, we may also project the tensor product to the spin-1 and spin-0 components first (r.h.s.).[12] The shift of the parameters u by $\pm\frac{i}{2}$ is a quantum effect. It is related to the fact that the tensor product only splits up when the parameters differ by i.

A generalisation of the above identity corresponding to $(\frac{1}{2}n) \otimes (\frac{1}{2}) = (\frac{1}{2}(n+1)) \oplus (\frac{1}{2}(n-1))$ reads

$$T_{n/2}(u+\tfrac{i}{2})T_{1/2}(u-\tfrac{i}{2}n) = \big(u-\tfrac{i}{2}(n-1)\big)^L T_{(n+1)/2}(u) + \big(u-\tfrac{i}{2}(n+1)\big)^L T_{(n-1)/2}(u+i). \tag{4.68}$$

This identity allows to recursively construct transfer matrix eigenvalues for representations with arbitrary spin

$$T_{n/2}(u) = \sum_{r=0}^{n} \big(u+\tfrac{i}{2}(n-2r)\big)^L \cdot \prod_{k=1}^{M} \frac{u-u_k+\frac{i}{2}(n+1)}{u-u_k+\frac{i}{2}(n+1-2r)} \frac{u-u_k-\frac{i}{2}(n+1)}{u-u_k+\frac{i}{2}(n-1-2r)} \,. \tag{4.69}$$

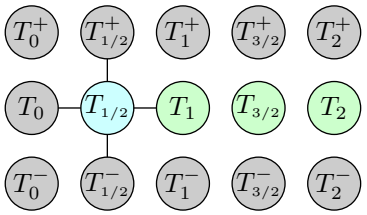
In particular for spin-0 we should set $T_0(u) = u^L$.

[12] Note that the product of two transfer matrices with rearranged connections due to the projections essentially winds twice around the closed chain. Therefore T_1 can be viewed as the analog of $F_2 = \operatorname{tr} L^2$ which takes two loops around the chain before closing (with suitable modifications for the quantum case.

There are many similar identities that relate the various transfer matrix eigenvalues. A very useful generalisation is the defining relation of the *T-system*[13]

$$T_{n/2}(u+\tfrac{i}{2})T_{n/2}(u-\tfrac{i}{2}) = T_{(n+1)/2}(u)T_{(n-1)/2}(u) + T^{+}_{n/2}(u)T^{-}_{n/2}(u). \tag{4.70}$$

This equation has the form of a difference equation reminiscent of the Hirota equation. It is defined on a lattice of points given by three rows T^{+}_{s}, T_s and T^{-}_{s}.



$\cdots$ (4.71)

In our case the top and bottom rows as well as the first site of the middle row are given by fixed boundary values

$$T^{\pm}_{n/2}(u) = \big(u \pm \tfrac{i}{2}(n+1)\big)^{L}, \qquad T_0(u) = u^L. \tag{4.72}$$

The middle row contains the dynamical information on the system. Requiring that all the $T_s(u)$ are polynomials is equivalent to the Bethe equation and determines the spectrum of the closed Heisenberg XXX spin chain.

The benefit of the T-system equation is that it generalises to much more complicated systems such as integrable quantum field theories and the so-called thermodynamic Bethe ansatz. To that end one has to set up a suitable lattice of functions[14] and specify appropriate boundary conditions. The drawback of this approach is that the T-system consists of infinitely many functions to be solved simultaneously.

[13] A reformulation of the T-system is the so-called *Y-system*. It uses a different set of variable functions Y to eliminate some unphysical degrees of freedom, but has a very similar form otherwise.

[14] The lattice is closely related to the symmetry algebra: The vertical direction corresponds to the Dynkin diagram; in our case there is just a single row (after removing the boundaries) corresponding to the single node of the Dynkin diagram for SU(2). The horizontal direction corresponds to a reduced n-fold symmetric product of the fundamental representation corresponding to one node of the Dynkin diagram; in our case this is the representation with spin $(n/2)$.

5 Integrable Statistical Mechanics

The R-matrix formalism opens up applications of integrability to specific models of statistical mechanics in 2 (discrete) dimensions. In the following we will sketch some basic concepts of these models and their solution.

5.1 Models of Statistical Mechanics

As for quantum mechanics, there exists a forest of models some of which have been studied extensively while others have been been invented. Let us present a few of these models which are relevant to the integrable context.

Ising Model. The Ising model is one of the most basic models of statistical mechanics. Is can be viewed as the statistical mechanics analog of the Heisenberg spin chain:

- it is based on two discrete spin values,
- interactions are typically between nearest neighbours,
- it describes magnetism,

The main distinction is that it is a statistical mechanics model rather than a quantum mechanical one.

Consider a lattice of spins. The spin σ_k at lattice site k can take two values, $+$ or $-$. A state of the model is an assignment of spins σ_k on all lattice sites.[1]

The energy of a state is given by a sum over all nearest neighbour pairs

$$E(\sigma) = -\lambda \sum_{(kl)} \sigma_k \sigma_l - h \sum_k \sigma_k. \tag{5.1}$$

The latter term describes the effect of a magnetic field which introduces a bias for the spin orientations. On the one hand one can now determine the minimum-energy configuration; for sufficiently large negative λ (compared to h) this would be an anti-ferromagnetic state with alternating spins, otherwise a ferromagnetic state with with all spins aligned (in the direction of the magnetic field).

The fundamental object in statistical mechanics is the partition function

$$Z(\beta;\lambda,h) = \sum_\sigma \exp\bigl(-\beta E(\sigma)\bigr), \tag{5.2}$$

where β denotes the inverse temperature. In the one-dimensional case, the problem has been solved by Ising. The two-dimensional case was solved by

[1] The quantum mechanical model would assign a (complex) number to each state.

Onsager based on the equivalence to lattice fermions. These two cases are particularly simple because they represent integrable models.

Let us briefly sketch the solution of the one-dimensional model by means of a transfer matrix. The contribution to Z of a pair of spins can be summarised in a 2×2 matrix V

$$V = \begin{pmatrix} e^{+\beta\lambda+\beta h} & e^{-\beta\lambda} \\ e^{-\beta\lambda} & e^{+\beta\lambda-\beta h} \end{pmatrix}. \tag{5.3}$$

Products of this matrix summarise the contribution from consecutive spins. Matrix multiplication takes care of the summation over intermediate spins. For a closed chain of L sites one therefore finds simply

$$Z = \operatorname{tr} V^L. \tag{5.4}$$

This expression can be evaluated by means of the eigenvalues of the above matrix.

This method is somewhat reminiscent of the methods used for integrable spin chains and we will see more of this at work later. Let us mention a relationship to a spin chain Hamiltonian here

$$\mathcal{H} = -\lambda \sum\nolimits_k \sigma_k^z \sigma_{k+1}^z - h \sum\nolimits_k \sigma_k^z. \tag{5.5}$$

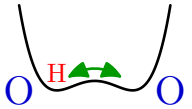
This is part of the XXZ family of spin chain Hamiltonians. It is a singular case because spin transport along the chain is frozen out. We can write the partition function as a trace over the space of states

$$Z = \operatorname{Tr} \exp(-\beta\mathcal{H}). \tag{5.6}$$

Note that the partition function tells us something about the complete spectrum of states rather than individual states.

Ice Model. The ice model is a model of the crystal structure of ice. Evidently, ice consists of water molecules H–O–H. These are arranged such that every oxygen atom is surrounded by 4 further oxygen atoms. On each of these links there is one hydrogen atom which is associated to either of the two oxygen atoms. Therefore there are two hydrogen atoms per oxygen atom in average.

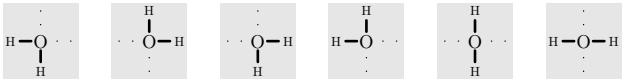
However, the structure of ice is slightly more elaborate: The potential for the hydrogen atoms has two minima, it can can reside in one of two spots along the line connecting the two adjacent oxygen atoms. On the other hand, the interactions between the atoms prefer configurations where two hydrogen atoms are close to each oxygen atom.



(5.7)

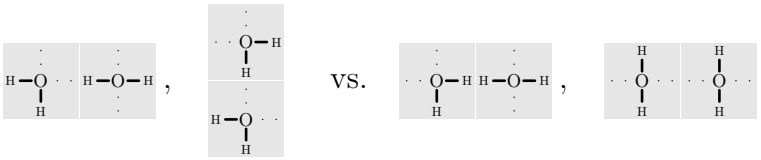
There are many configurations satisfying these criteria. To understand the entropy of ice, one has to count such configurations.

The model is a prototype of the class of *vertex models*. In distinction to the Ising model,[2] the fundamental building block is a vertex which can be in one of several configurations.



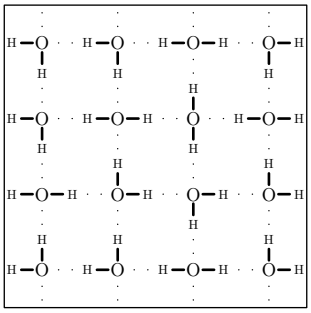
(5.8)

Adjacent vertices have to satisfy certain compatibility conditions. In our case, each link between two atoms has to be singly occupied. Two allowed and two disallowed junctions are



(5.9)

The structure of ice is three-dimensional. For a realistic one would have to use a tetrahedronal structure as the adjacency information. As a more abstract model, one can use a two-dimensional square lattice.



(5.10)

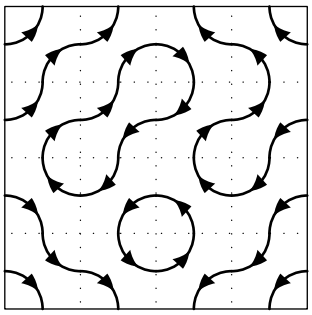
As one can see, many different configurations of this type are conceivable.

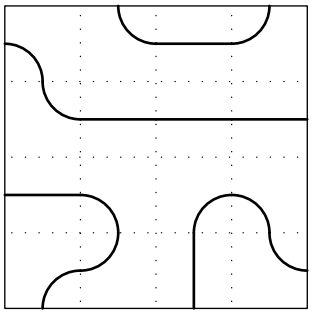
Lattice Path Models. A seemingly different class of models are *lattice path models*. Here one starts with a lattice. Paths are drawn on this lattice according to a particular set of rules, e.g. paths may or may not

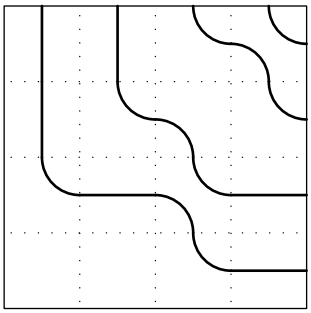
- form loops,
- be allowed to cross,
- be allowed to have straight segments or certain types of curves,
- be directed,
- fill all available space,
- . . . .

[2]Evidently, the Ising model with interactions between nearest neighbours can be represented as a vertex model. The above construction in terms of the matrix V such an implementation.

Examples of states in three different lattice path models are:

CPL:  (5.11)

FPL:  (5.12)

DLP:  (5.13)

Alternating Sign Matrices. An old problem of combinatorics is *alternating sign matrices*. On each row and on each column these matrices

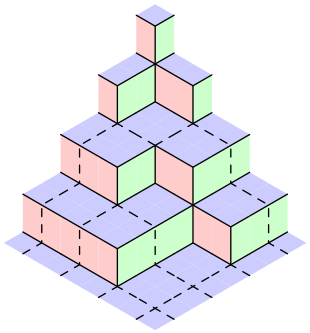
- have an alternating sequence of $+1$'s and -1's
- which starts and ends with $+1$ and
- is diluted by an arbitrary number of of 0's.

An example of a 4×4 alternating sign matrix is

$$\begin{pmatrix} 0 & 0 & + & 0 \\ 0 & + & - & + \\ + & 0 & 0 & 0 \\ 0 & 0 & + & 0 \end{pmatrix} . \tag{5.14}$$

The number of alternating sign matrices is a rapidly increasing sequence starting as $1, 2, 7, 42, 429, \ldots$.

Box Storage Models. A final class of models is concerned with stacking boxes in the corner of a room, e.g.:



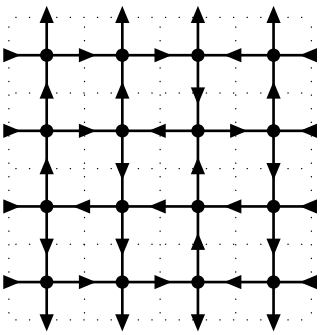
(5.15)

The stack of boxes must not decrease when moving closer towards the corner of the room. Here one may or may not restrict the increase of height by one box per unit step.

Note that this model is evidently equivalent to a rhombus (lozenge) tiling problem. The latter is equivalent to the dimer problem on the honeycomb lattice which is relevant to graphene. One can also relate the model with maximum step size to one of the lattice path models (DLP) where the latter represents the height contours of the former.

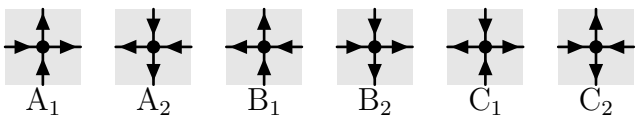
Six-Vertex Model. Most of the above models are particular formulations of the six-vertex model:

- The six-vertex model is a vertex model consisting of 6 types of vertices.
- Each vertex has 4 neighbours.
- Two neighbouring vertices are joined by a directed line.
- At each vertex there must be precisely two ingoing and two outgoing lines.



(5.16)

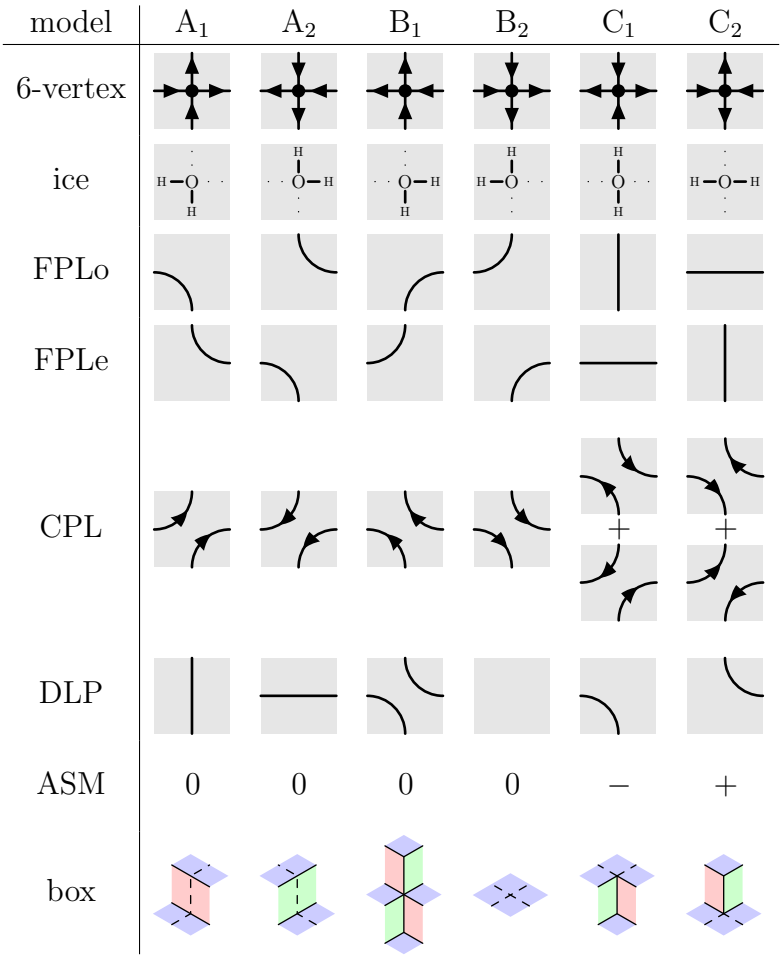
The 6 vertices are usually denoted by $ABC_{1,2}$:



(5.17)

By decomposing the space of the above models into a lattice of cells, we find that all of them are vertex models and there exists the following dictionary for the

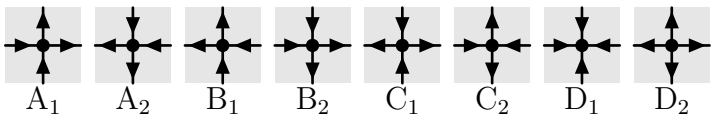
vertices:



(5.18)

Note that the correspondence is not direct for two of the lattice path models. The model denoted by FPL requires separate dictionaries for the even and odd cells of the lattice, respectively. The model denoted by CPL in fact has 8 vertices, and two pairs of vertices (with the same outer links, but different internal connections) are encoded by two single vertices in the 6-vertex model. We will see later how this situation can be interpreted.

A generalisation of the 6-vertex model is the 8-vertex model which has two additional vertices with four ingoing or four outgoing lines:



(5.19)

It is more general in the sense that it has sources and sinks for the flow which violate the conservation of the flow of arrows. Many vertex models with more vertices have been considered. Often they are denoted by the number of vertices, e.g. a nineteen-vertex model.

5.2 Integrability

Boltzmann Weights. For combinatorial models, it is usually sufficient to count the number of permissible configurations. In statistical mechanics one may in addition want to compute the partition function at a given inverse temperature β. In that case, each of the 6 vertices is attributed a certain energy e_v. The partition function is given by

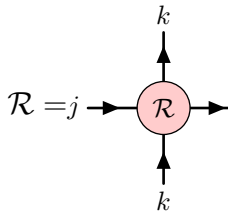
$$Z = \sum_{v(k)} \exp(-\beta E), \qquad E = \sum_k e_{v(k)}. \tag{5.20}$$

The partition function can also be written as a sum of products of *Boltzmann weights* $\exp(-\beta e_{v(k)})$ for the vertices

$$Z = \sum_{v(k)} \prod_k P_{v(k)}, \qquad P_v = \exp\bigl(-\beta e_v\bigr). \tag{5.21}$$

The Boltzmann weights of the 6 vertices ABC_{12} are denoted by a_{12}, b_{12}, c_{12}, respectively. The configuration of the above sample state of the 6 vertex model contributes the term $a_1^2 a_2^2 b_1^3 b_2^3 c_1^5 c_2^1$ to the partition function.

It makes sense to collect the Boltzmann weights into a matrix $\mathcal{R}$

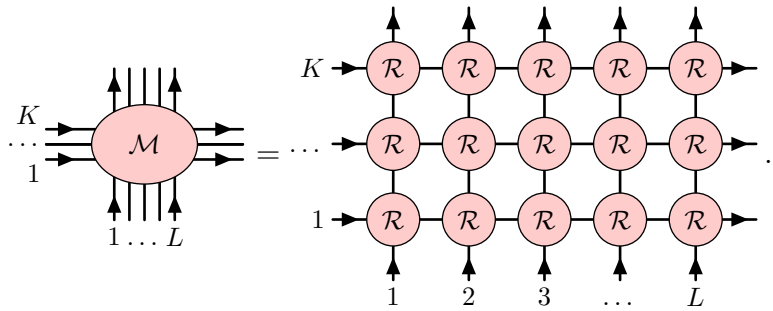
$\mathcal{R} =$ 

$= \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & b_1 & c_1 & 0 \\ 0 & c_2 & b_2 & 0 \\ 0 & 0 & 0 & a_2 \end{pmatrix}$

$= a_1 \left|\uparrow\rightarrow\right\rangle\left\langle\rightarrow\uparrow\right| + b_1 \left|\uparrow\leftarrow\right\rangle\left\langle\leftarrow\uparrow\right| + c_1 \left|\downarrow\rightarrow\right\rangle\left\langle\leftarrow\uparrow\right|$

$\quad + c_2 \left|\downarrow\leftarrow\right\rangle\left\langle\rightarrow\downarrow\right| + b_2 \left|\downarrow\rightarrow\right\rangle\left\langle\rightarrow\downarrow\right| + a_2 \left|\downarrow\leftarrow\right\rangle\left\langle\leftarrow\downarrow\right|.$ (5.22)

In order to compute the partition function for a lattice of size $L \times K$, all we have to do is to multiply these matrices appropriately in a big lattice matrix $\mathcal{M}$. This is described conveniently in the graphical notation we introduced earlier:



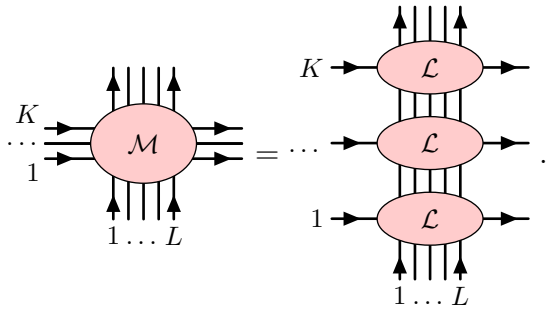
. (5.23)

The lattice matrix essentially describes the partition function. The matrix $\mathcal{R}$ sums over all possible vertices with corresponding Boltzmann weight. Matrix multiplication then takes care that only matching adjacent vertices are selected.

In order to compute the lattice matrix it is convenient to decompose the lattice matrix $\mathcal{M}$ into row matrices $\mathcal{L}_j$ as

$$\mathcal{M} = \mathcal{L}_K \mathcal{L}_{K-1} \dots \mathcal{L}_2 \mathcal{L}_1, \tag{5.24}$$

or in graphical notation

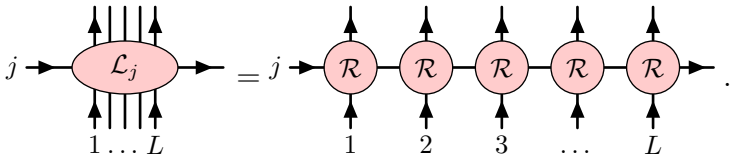


(5.25)

The row matrices summarise the contribution of a single row j of vertices

$$\mathcal{L}_j = \mathcal{R}_{j,L} \mathcal{R}_{j,L-1} \dots \mathcal{R}_{j,2} \mathcal{R}_{j,1}. \tag{5.26}$$

Note that this is not to be understood as a standard matrix product of the above matrices. The above matrix $\mathcal{R}$ is in fact a tensor of rank 4 and each product merely multiplies along one of the components spaces



(5.27)

Alternatively, one could decompose the lattice matrix into column matrices $\tilde{\mathcal{L}}$.

So far we have not taken boundary conditions into account. The lattice matrix becomes the partition function after the boundary conditions are implemented appropriately. We shall discuss two relevant boundary conditions further below.

Integrable R-Matrix. We are observing a close similarity to the R-matrix framework for integrable models. The R-matrix of the 6-vertex model has the same form as the R-matrix for the Heisenberg XXZ chain[3]

$$\mathcal{R}(x, y; q) \sim \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & b_1 & c_1 & 0 \\ 0 & c_2 & b_2 & 0 \\ 0 & 0 & 0 & a_2 \end{pmatrix}. \tag{5.28}$$

The entries of the R-matrix should therefore be interpreted as Boltzmann weights in the statistical mechanics context. The overall scaling of the R-matrix elements

[3]Note that the R-matrix of the 8-vertex model corresponds to the Heisenberg XYZ chain. Conversely, the Heisenberg XXX chain corresponds to a special case of the 6-vertex model without a reduction of the number of vertices.

is largely irrelevant, and we can adjust it at will. We adjust it such that the coefficients are Laurent polynomials in the parameters.

The coefficients of the R-matrix of the XXZ model take the standard form

$$\begin{aligned} a &= (x/y)q - (y/x)/q, \\ b &= (x/y) - (y/x), \\ c &= q - 1/q. \end{aligned} \tag{5.29}$$

Here x and y are the parameters associated to the two contributing vector spaces.[4] The deformation parameter q is a global parameter of the XXZ model.[5] Note that the vertices with opposite directions of all arrows commonly take the same values, i.e.

$$a_1 = a_2 = a, \qquad b_1 = b_2 = b, \qquad c_1 = c_2 = c. \tag{5.30}$$

However, in some situations one needs more general weights compatible with the constraints

$$a_1 a_2 = a^2, \qquad b_1 b_2 = b^2, \qquad c_1 c_2 = c^2. \tag{5.31}$$

Most frequently, this generalisation is applied to accommodate for a $c_1 \neq c_2$. This generalisation can always be achieved by an adjustment of boundary conditions, and therefore it is without physical consequences. Generalisations of the type $a_1 \neq a_2$ or $b_1 \neq b_2$ typically have some impact on physics. Note that for the 6-vertex model we are rather free to choose the Boltzmann weights independently while preserving integrability. For higher-vertex vertex models, the configuration of Boltzmann weights for integrable models is very restricted.

The parameters x_j and y_k can be chosen individually per row and per column, respectively. For homogeneous models, however, one would typically choose them to be all equal. In this case the Boltzmann weights are independent of the location of the vertex. Nevertheless, one should allow x and y to take independent values. Furthermore, it may be desirable to have a rotational symmetry for the R-matrix. This is achieved by setting $a = b$.

Parameter Values. In order to investigate the coefficients a, b, c more conveniently, let us choose an overall normalisation such that $b = 1$. Furthermore, split c_1 and c_2 according to

$$\frac{c_1}{c_2} = -q\,\frac{x^2}{y^2}\,. \tag{5.32}$$

Finally solve x/y for a

$$\frac{x}{y} = \sqrt{\frac{a - 1/q}{a - q}}\,. \tag{5.33}$$

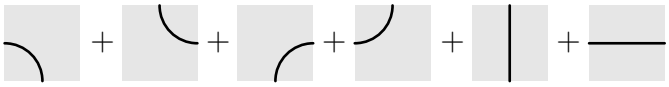
[4]The R-matrix is written in a quotient form. The difference form is obtained by setting $x, y = x_0 \exp(u, v)$.

[5]Other parametrisations involve the parameters $\Delta = \frac{1}{2}(q + 1/q)$ and $q = \exp(\hbar)$ or $q = \exp(i\hbar)$. The R-matrix for the Heisenberg XXX model is recovered for $q \to 1$ with $x, y = q^{u,v}$ and a suitable rescaling of $\mathcal{R}$.

This leads to the following set of Boltzmann weights

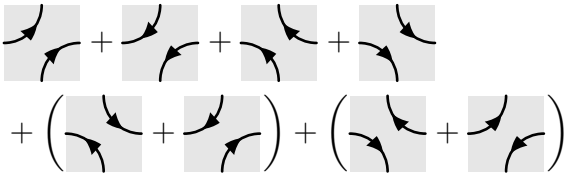
$$b = 1, \qquad c_1 = iq^{-1/2} - iaq^{+1/2}, \qquad c_2 = iaq^{-1/2} - iq^{+1/2}. \tag{5.34}$$

Let us now consider two of the lattice path models. For the first model (FPL) we would like to have equal weights for all vertices

$\mathcal{R} =$  . (5.35)

This is achieved by setting $a = 1$ and $q = \exp(\pm i\pi/3)$ corresponding to $\Delta = \frac{1}{2}$.

For the second model we have the directed paths

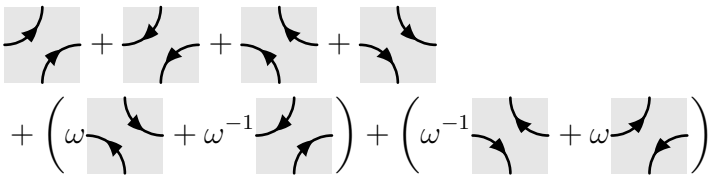
$\mathcal{R} =$  (5.36)

Here we need $a = 1$ for the same reason as above. Now the two vertices c are presented by the four lattice path configurations on the second line. Since these have equivalent links to adjacent vertices, we must set $c_1 = c_2 = 2$. This is achieved by $q = -1$ or $\Delta = -1$.

Note that there is a useful generalisation of the previous model if we keep $q^{1/2} = i\omega$ unspecified. Then the coefficients read

$$a = b = 1, \qquad c_1 = c_2 = \omega + \omega^{-1}. \tag{5.37}$$

and we can write the R-matrix as

$\mathcal{R} =$  (5.38)

Here the two terms $\omega^{\pm}$ have been assigned to the two lattice path configurations which contribute to the counting in the same way. Any other distribution leading to the same sum would be equally permissible. This distribution, however, is distinguished because the power of ω is related to the turning number of the paths: For each quarter turn towards the left or right there is a factor of $\omega^{1/2}$ and $\omega^{-1/2}$, respectively. The overall turning number of the first four path configurations is zero, but for the latter four it is half turn in either direction. One can keep track of these factor in the partition function

$$Z = \sum_{k=-\infty}^{\infty} \omega^k Z_k. \tag{5.39}$$

Then Z_k measures the contributions of loops with total turning number $k/2$ towards the left.

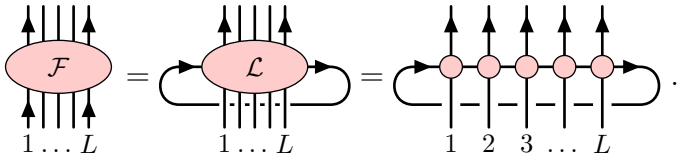
By adjusting the Boltzmann weights appropriately, one can try to measure different quantities of the configurations such as the number of loops or the number of self-interactions. One could also also use the specific choice $\omega^4 = -1$ to suppress configurations with loops altogether because for every clockwise loop there is a counterclockwise loop with the negative weight.

Periodic Boundary Conditions. We have not yet specified boundary conditions. A convenient choice is periodic boundary conditions in one or in both directions.

If one chooses the horizontal direction to be periodic, the row matrix can be turned into a row transfer matrix by a trace (potentially after inserting a twist matrix)

$$\mathcal{F} = \mathrm{tr}_j \, \mathcal{L}_j \tag{5.40}$$

or in figures



(5.41)

When also the vertical direction is periodic, the partition function is given by

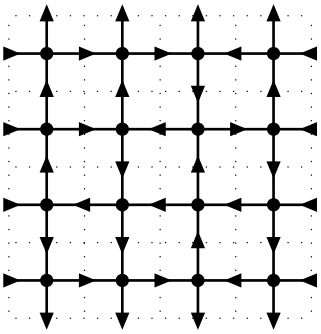
$$Z = \mathrm{Tr} \, \mathcal{F}^K. \tag{5.42}$$

Therefore, the partition function is determined by the eigenvalue spectrum of the row transfer matrix. In particular, this leads to useful approximations for a very long lattice in the vertical direction. In this case, the largest eigenvalues yield the dominant contributions. The former correspond to the lowest-energy configurations, so this statement makes perfect physical sense, and it allows to derive more concrete statements.

The techniques of quantum integrable systems can now be applied to the system. Here it often makes sense to keep the values of the parameters x_k, y_j arbitrary during the calculation. This allows to investigate the analytical dependence of the observables on them. After having gained a good understanding of the analytical behaviour, one can use it towards construction of the answer. In the answer one can then adjust the parameters to the desired values.

Domain Wall Boundary Conditions. Another boundary condition that has been heavily investigates is domain wall boundary conditions. Here one restricts to a square lattice of size L. All the external links of the vertices are forced in equal configurations along each side of the square. In the 6-vertex description, the horizontal external arrows all point inwards. Consequently, the vertical arrows must point outwards in order to have a conserved flow through the lattice. An

example is given by the configuration:



(5.43)

This partition function for this problem is described by the $\mathcal{B}$ element of the row matrix $\mathcal{L}$ discussed earlier in the context of the algebraic Bethe ansatz. The partition function takes the form

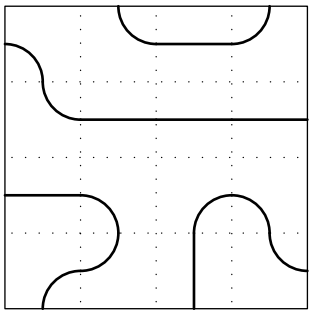
$$Z = \langle \bar{0} | \mathcal{B}^L | 0 \rangle. \tag{5.44}$$

This problem can again be attacked by quantum integrable methods. The result is reminiscent of a determinant formula.

This set of boundary conditions is relevant to alternating sign matrices as one can easily convince oneself. For example, the above pattern corresponds to the alternating sign matrix

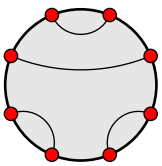
$$\begin{pmatrix} 0 & 0 & + & 0 \\ 0 & + & - & + \\ + & 0 & 0 & 0 \\ 0 & 0 & + & 0 \end{pmatrix}. \tag{5.45}$$

Razumov–Stroganov Duality. We are now in the position to introduce the Razumov–Strogonov duality which is a curious relationship between two different lattice path models: On the one hand, there is the FPL model with domain wall boundary conditions on an $L \times L$ square. Its configurations



(5.46)

can be viewed as link patterns on the disk. A link pattern is a configuration of lines which connect $2L$ marked point on the boundary without crossing.



(5.47)

On the other hard, there is the CPL model on the semi-infinite cylinder of circumference $2L$. The states of this model also connect the boundaries according to a link pattern.[6] The duality relates the probability of finding a state of the CPL model with a given link pattern to the number of states of the FPL model with the same link pattern. Note that the former can be addressed by the wave function of the ground state of the transfer matrix $\mathcal{F}$.

[6]Paths starting at the boundary and ending at infinity are suppressed.

6 Quantum Algebra

Integrability can be viewed a hidden extended symmetry of a model. In the following we will discuss several symmetry groups and concepts that come to play in this context.

6.1 Lie Algebra

Continuous symmetries in physics are usually describes by Lie algebras. Here we introduce some elements of Lie theory that come to use in integrable systems.

Lie Algebras. We assume familiarity with the concepts of Lie algebra. Nevertheless, let us give a summary of the most important features:

- A Lie algebra is a vector space $\mathfrak{g}$ equipped with Lie brackets

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}. \tag{6.1}$$

  Lie brackets are bi-linear, anti-symmetric and which satisfy the Jacobi identities.
- We will assume the algebras to be complex[1] and simple. Integrability is largely related to infinite-dimensional algebras which in turn are based on finite-dimensional algebras.
- A representation of a Lie algebra on a vector space $\mathbb{V}$ is a linear map

$$\rho : \mathfrak{g} \to \mathrm{End}(\mathbb{V}), \tag{6.2}$$

  which preserves the Lie brackets as commutators on $\mathrm{End}(\mathbb{V})$

$$\big[\rho(x),\rho(y)\big] = \rho\big([x,y]\big). \tag{6.3}$$

- We will often use a basis for J^a, $a = 1,\ldots,\dim(\mathfrak{g})$, for the space $\mathfrak{g}$.
- The Lie brackets are encoded by the structure constants f_c^{ab}

$$[\mathrm{J}^a,\mathrm{J}^b] = f_c^{ab}\mathrm{J}^c. \tag{6.4}$$

- We usually have an invariant symmetric quadratic form

$$T = c_{ab}\mathrm{J}^a \otimes \mathrm{J}^b, \tag{6.5}$$

  which is the inverse the of Cartan–Killing form

$$K(x,y) = \mathrm{tr}\,\rho_{\mathrm{ad}}(x)\rho_{\mathrm{ad}}(y), \qquad c^{ab} \sim K(\mathrm{J}^a,\mathrm{J}^b). \tag{6.6}$$

[1] Real algebras are equally suitable, but require additional care.

Loop Algebras. The algebras of integrability for field theories are typically based on infinite-dimensional loop algebras. The *loop algebra* $\mathfrak{g}[u, u^{-1}]$ [2] is an infinite-dimensional Lie algebra:

- It is based on some finite-dimensional Lie algebra $\mathfrak{g}$.
- It is spanned by the elements $\mathrm{J}^a_n := u^n \otimes \mathrm{J}^a$ with $a = 1, \ldots, \dim(g)$ and $n \in \mathbb{Z}$.[3] The integer n can be called the *level* of the element.
- The Lie brackets of the loop algebra are defined in terms of the Lie brackets of $\mathfrak{g}$ as
$$[\mathrm{J}^a_m, \mathrm{J}^b_n] = f^{ab}_c \mathrm{J}^c_{m+n}. \tag{6.7}$$
Evidently, the Lie brackets satisfy the Jacobi identity.
- The subalgebra at level $n = 0$ is the original Lie algebra $\mathfrak{g}$.[4]
- There is a tower of quadratic invariant forms
$$\hat{T}_n = \sum_{k=-\infty}^{\infty} c_{ab} \mathrm{J}^a_k \otimes \mathrm{J}^b_{n-k}. \tag{6.8}$$

Another relevant class of algebras are polynomial or *half loop algebras*. These are loop algebras restricted to positive or negative levels n. The zeroth level may or may not be included. For positive levels, the algebras are denoted by $\mathfrak{g}[u]$ or $u\mathfrak{g}[u]$ depending on whether the zeroth level is included or not.

For integrable systems the class of *evaluation representations* is very important. For a given representation ρ of a finite-dimensional Lie algebra $\mathfrak{g}$ on the space $\mathbb{V}$ it is straight-forward to construct a corresponding one-parameter family of representations ρ_u of a (half) loop algebra
$$\rho_u(\mathrm{J}^a_n) := u^n \rho(\mathrm{J}^a). \tag{6.9}$$
The constant $u \in \mathbb{C}$ of the representation ρ_u is called the *evaluation parameter*.

A useful feature is that the evaluation representation has the same dimension as the underlying representation of the finite-dimensional Lie algebra. In particular, it can be finite. Moreover, the tensor product $\rho_{u,v} = \rho_u \otimes 1 + 1 \otimes \rho_v$ of two evaluation representations ρ_u, ρ_v is irreducible unless the evaluation parameters match, $u = v$. This has strong implications on invariant objects.

Affine Kac–Moody Algebras. Finally, let us mention the *affine Kac–Moody algebra* $\hat{\mathfrak{g}}$. This is the loop algebra $\mathfrak{g}[u, u^{-1}]$ extended by a *central element* C which arises in the Lie brackets
$$[\mathrm{J}^a_m, \mathrm{J}^b_n] = f^{ab}_c \mathrm{J}^c_{m+n} + m\delta_{m+n,0} c^{ab} \mathrm{C}. \tag{6.10}$$

[2] Here, we will not make a thorough distinction between polynomial algebras, their completion and formal power series.

[3] A loop algebra is formally defined by maps from the circle S^1 ("loop") to $\mathfrak{g}$. To see the relationship, set $u = \exp(i\varphi)$ and perform a Fourier expansion in φ.

[4] A useful fact to keep in mind is that the original Lie algebra can be embedded into the loop algebra in many ways: Given a $\mathbb{Z}$-grading (generated by some element of the Cartan algebra), one can identify the level with (a multiple of) this grading and obtain the same Lie algebra.

Sometimes a *derivation* element D is also included in the affine algebra

$$[\mathrm{D}, \mathrm{J}_n^a] = n\mathrm{J}_n^a. \tag{6.11}$$

It serves as the conjugate of the central element C. It appears in the quadratic invariant form $\hat{T}_0$ of $\hat{\mathfrak{g}}$ in the combination $\mathrm{B} \otimes \mathrm{C} + \mathrm{C} \otimes \mathrm{B}$ and makes it invertible.

For our purposes, it makes sense to define a different derivation[5] [6]

$$[\mathrm{D}, \mathrm{J}_n^a] = n\mathrm{J}_{n-1}^a. \tag{6.12}$$

The relevant quadratic invariant form for this algebra is $\hat{T}_{-1}$.

Evaluation representations also exist for affine algebras, where they have vanishing central element eigenvalue. The presence of a derivation changes the situation: it acts on the evaluation parameter as a derivative. Then only the family of evaluation representations forms a representation of the enlarged algebra. This representation can be viewed as a one-dimensional *field* where the derivation acts as the momentum generator. The enlarged algebra thus covers spacetime symmetries of a 2-dimensional field theory.

Loop algebras are subalgebras of the affine Kac–Moody algebras where the central element has been projected out (and where the derivation has been dropped). In the following we will mostly consider loop algebras keeping in mind that the discussions could be extended to affine Kac–Moody algebras with minor adjustments.

6.2 Classical Integrability

In classical integrability we have derived a classical r-matrix satisfying the classical Yang–Baxter equation. A classical r-matrix fits well into the framework of *Lie bialgebras*.

Lie Bialgebra. A Lie bialgebra is a Lie algebra $\mathfrak{g}$ whose dual $\mathfrak{g}^*$ is also a Lie algebra such that the two Lie brackets are compatible.

It is convenient to formulate the above statements purely in terms of $\mathfrak{g}$ without reference to the dual $\mathfrak{g}^*$. To that end, let us discuss the dual of a Lie bracket: Define an operation $\mu^* : \mathfrak{g}^* \to \mathfrak{g}^* \otimes \mathfrak{g}^*$ such that for all $x, y \in \mathfrak{g}$ and $c \in \mathfrak{g}^*$

$$c([x, y]) = \mu^*(c)(x \otimes y). \tag{6.13}$$

Conversely, the dual of the dual Lie bracket, the so-called *Lie cobracket* δ, is defined as a linear map

$$\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}. \tag{6.14}$$

[5] One can transform between the two forms of the derivation by an exponential map $z = \exp(u)$ such that $zd/dz = d/du$.

[6] In fact, the derivation could be extended to a Virasoro algebra, but we need merely one additional element serving as the conjugate to C.

The cobracket must return an anti-symmetric element of the tensor product of two Lie algebras. It must also satisfy the dual of the Jacobi identity for all $c \in \mathfrak{g}$

$$\big(1 + \mathcal{P}_{12}\mathcal{P}_{23} + \mathcal{P}_{23}\mathcal{P}_{12}\big)(\delta \otimes 1)\big(\delta(c)\big) = 0. \tag{6.15}$$

Compatibility between the algebra and the coalgebra is the statement

$$\delta\big([x, y]\big) = \big[x, \delta(y)\big] + \big[\delta(x), y\big], \tag{6.16}$$

where the Lie bracket on the tensor product is defined as

$$[x, y \otimes z] := [x, y] \otimes z + y \otimes [x, z], \tag{6.17}$$

and similarly for the other combination. The role of the cobracket will become clearer in the context of quantum algebras to be discussed below.

Classical r-Matrix. The classical r-matrix in the algebraic context is an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ such that

$$\delta(x) = [r, x]. \tag{6.18}$$

Anti-symmetry of δ requires that the symmetric part $r + \mathcal{P}(r)$ is an invariant element of $\mathfrak{g} \otimes \mathfrak{g}$ (essentially proportional the invariant quadratic form). Furthermore, the dual Jacobi identity and the compatibility condition requires that the combination

$$[[r, r]] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \tag{6.19}$$

is an invariant element of $\mathfrak{g}^{\otimes 3}$.

- A Lie bialgebra with r-matrix is called *coboundary*.
- A *coboundary* Lie bialgebra is called *quasi-triangular* if the classical Yang–Baxter equation holds
$$[[r, r]] = 0. \tag{6.20}$$
- A *quasi-triangular* Lie bialgebra is called *triangular* if the r-matrix is anti-symmetric
$$r = -\mathcal{P}(r). \tag{6.21}$$

Example. Earlier we have discussed a classical r-matrix of the form[7]

$$r(u, v) = \frac{c_{ab}\mathrm{J}^a \otimes \mathrm{J}^b}{u - v} = \frac{T}{u - v}\,. \tag{6.22}$$

We can recast it into an element of a loop algebra

$$r = \frac{c_{ab}\mathrm{J}^a \otimes \mathrm{J}^b}{u - v} \in \mathfrak{g}[u, u^{-1}] \otimes \mathfrak{g}[v, v^{-1}]. \tag{6.23}$$

[7]More precisely, we discussed a representation $(\rho \otimes \rho)(r)$ of this r-matrix. In order to match with the below forms of the r-matrix as elements of loop algebras one employs evaluation representations $(\rho_u \otimes \rho_v)(r)$.

It makes sense to expand this expression into levels by means of a formal power series. Assuming that $|u| \gg |v|$ we find

$$r = \sum_{k=0}^{\infty} \frac{v^n}{u^{n+1}}\, T = \sum_{k=0}^{\infty} c_{ab} \mathrm{J}^a_{-n-1} \otimes \mathrm{J}^b_n. \tag{6.24}$$

This r-matrix satisfies the classical Yang–Baxter equation $[[r, r]] = 0$. Note, however, that it is not anti-symmetric as the above rational form suggests.[8] Nevertheless, the symmetric part of r is invariant as desired

$$r + \mathcal{P}(r) = \sum_{k=-\infty}^{\infty} c_{ab} \mathrm{J}^a_{-n-1} \otimes \mathrm{J}^b_n = \hat{T}_{-1}. \tag{6.25}$$

Therefore this r-matrix describes a quasi-triangular Lie bialgebra.

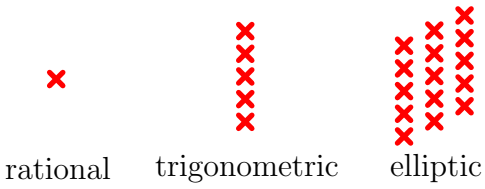
Alternatively, we can perform an expansion with $|x| \ll |y|$

$$\tilde{r} = -\sum_{k=0}^{\infty} \frac{u^n}{v^{n+1}}\, T = -\sum_{k=0}^{\infty} c_{ab} \mathrm{J}^a_n \otimes \mathrm{J}^b_{-n-1}. \tag{6.26}$$

Likewise, this r-matrix satisfies the classical Yang–Baxter equation, and we find the symmetric part $r + \mathcal{P}(r) = -\hat{T}_{-1}$.

It is tempting to take the linear combination $r' = r + \tilde{r}$ to remove the symmetric part of r'. Unfortunately, this r-matrix *does not* satisfy the classical Yang–Baxter equation $[[r + \tilde{r}, r + \tilde{r}]] \neq 0$ essentially because the latter is a non-linear relationship and therefore may change under linear combinations.

Classification and Construction. Solutions to the classical Yang–Baxter equation have been studied to some extent. In particular, the solutions of difference form for simple Lie algebras have been classified by Belavin and Drinfeld. There are essentially three classes depending on the location of poles in the complex plane:



(6.27)

- rational solutions with a single pole,
- trigonometric solutions with a one-dimensional lattice of poles,
- elliptic solutions with a two-dimensional lattice of poles.

[8] The crucial point is that the quadratic invariant $\hat{T}_{-1}$ is zero almost everywhere, but it is not identically zero. In accordance with Fourier transformations, it could be viewed as a delta function $\hat{T}_{-1} \sim T\delta(x - y)$.

For quantum integrable systems these three cases correspond to the Heisenberg XXX, XXZ and XYZ models, respectively.

Towards a construction of r-matrices, a useful observation in the (first version of the) above example is that the r-matrix belongs to the space

$$r \in u^{-1}\mathfrak{g}[u^{-1}] \otimes \mathfrak{g}[v]. \tag{6.28}$$

Importantly, the space on the left is the conjugate of the space on the right with respect to the quadratic form $\hat{T}_{-1}$. Therefore r as a matrix has a triangular form.

There is a construction which leads to r-matrices of a similar form. Starting with a conventional Lie bialgebra $\mathfrak{g}$, one can construct the *classical double algebra* $\mathfrak{dg} = \mathfrak{g} \oplus \mathfrak{g}^*$. Interestingly, the double algebra has a natural quasi-triangular structure. It also has the structure of a *Manin triple* $(\mathfrak{dg}, \mathfrak{g}, \mathfrak{g}^*)$. In our example, the starting point is the half loop algebra $\mathfrak{g}[u]$. Its dual is $\mathfrak{g}[u]^* = u^{-1}\mathfrak{g}[u^{-1}]$ and the double algebra is the complete loop algebra $\mathfrak{dg}[u] = \mathfrak{g}[u, u^{-1}]$.

6.3 Quantum Algebras

The symmetries of integrable quantum models are typically encoded into so-called *quantum algebras* based on loop and affine Lie algebras. Next we present some basic elements of quantum algebra.

Enveloping Algebra. In quantum physics one typically considers neither Lie groups G nor Lie algebras $\mathfrak{g}$, but rather their enveloping algebra $\mathrm{U}(\mathfrak{g})$.

Towards defining enveloping algebras, consider first the tensor algebra $\mathrm{T}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. The elements of this algebra are polynomials in the elements of $\mathfrak{g}$ which are assumed not to commute within monomials. Multiplication within the tensor algebra is defined by concatenation of monomials. The tensor algebra merely inherits the vector space $\mathfrak{g}$ of the Lie algebra, but not its algebraic structure.

The enveloping algebra $\mathrm{U}(\mathfrak{g})$ is obtained by identifying commutators of elements of $\mathfrak{g}$ with the corresponding Lie bracket

$$\mathrm{J}^a\mathrm{J}^b - \mathrm{J}^b\mathrm{J}^a = [\mathrm{J}^a, \mathrm{J}^b] = f_c^{ab}\mathrm{J}^c. \tag{6.29}$$

Alternatively one can define the enveloping algebra as a quotient of the tensor algebra by the ideal spanned by the commutation relations

$$\mathrm{U}[\mathfrak{g}] = \mathrm{T}[\mathfrak{g}] \,/\, \mathrm{span}(\mathrm{J}^a\mathrm{J}^b - \mathrm{J}^b\mathrm{J}^a - f_c^{ab}\mathrm{J}^c). \tag{6.30}$$

This identification implies that monomials of J^a can be reordered arbitrarily at the cost of shorter polynomials. A basis of $\mathrm{U}(\mathfrak{g})$ is therefore formed by orderless monomials in the J^a.[9]

In the context of quantum physics, an enveloping algebra has several advantages over plain Lie groups and algebras:

[9] The ordering of the letters matter for the algebraic structure, but not for enumerating a basis for the space of the algebra.

- It incorporates the Lie algebra $g = \mathrm{span}(\mathrm{J}^a)$ as the single-letter words and via the commutation relations.
- It incorporates operator products $\mathrm{J}^a\mathrm{J}^b$ which are essential for quantum mechanics.
- It incorporates the Lie group (formally) via the exponential map $\mathrm{G} = \{\exp(x), x \in \mathfrak{g}\}$.
- Tensor products of representations are naturally defined.
- It allows for non-trivial deformations which come to use in integrable systems.

Hopf Algebra. The enveloping algebra has a natural *Hopf algebra* structure. A Hopf algebra is a (co)unital (co)associative bialgebra with an antipode map. Let us summarise the various properties of a Hopf algebra A over a field $\mathbb{K}$:

- The product μ and coproduct Δ are $\mathbb{K}$-linear (co)associative maps

$$\mu : \mathrm{A} \otimes \mathrm{A} \to \mathrm{A}, \qquad \Delta : \mathrm{A} \to \mathrm{A} \otimes \mathrm{A}, \tag{6.31}$$

  which are compatible in the following sense (for $X, Y \in \mathrm{A}$)

$$\Delta\big(\mu(X \otimes Y)\big) = (\mu_{13} \otimes \mu_{24})\big(\Delta(X) \otimes \Delta(Y)\big). \tag{6.32}$$

  Note that the compatibility relation ensures that tensor product representations are consistently defined via the coproduct

$$\rho_{12}(X) := (\rho_1 \otimes \rho_2)\big(\Delta(X)\big). \tag{6.33}$$

- The unit ϵ and counit η formalise the existence of a unit element $1 = \epsilon(1)$

$$\epsilon : \mathbb{K} \to \mathrm{A}, \qquad \eta : \mathrm{A} \to \mathbb{K}. \tag{6.34}$$

  They must satisfy the usual compatibility relations (for $x \in \mathbb{K}$, $Y \in \mathrm{A}$)

$$\mu\big(\epsilon(x) \otimes Y\big) = xY, \qquad \eta_1\big(\Delta(X)\big) = X. \tag{6.35}$$

- The antipode Σ is a linear map on the algebra

$$\Sigma : \mathrm{A} \to \mathrm{A}, \tag{6.36}$$

  which satisfies

$$\mu\big(\Sigma_1(\Delta(X))\big) = \eta\big(\epsilon(X)\big). \tag{6.37}$$

  If an antipode exists for a bialgebra, it is unique. Furthermore, the antipode is an anti-homomorphism of the algebra and of the coalgebra

$$\mu\big(\Sigma(X) \otimes \Sigma(Y)\big) = \Sigma\big(\mu(Y \otimes X)\big), \tag{6.38}$$

$$\Delta\big(\Sigma(X)\big) = (\Sigma \otimes \Sigma)\big(\tilde{\Delta}(X)\big). \tag{6.39}$$

  Here $\tilde{\Delta}$ denotes the opposite coproduct with the two tensor factors interchanged.

We illustrate the meaning of the maps using the example of the enveloping algebra U[$\mathfrak{g}$]. The product is defined by concatenation of monomials

$$\mu(X \otimes Y) := XY. \tag{6.40}$$

Note that the algebraic relations of $\mathfrak{g}$ are implemented by identifications among the elements.

The coproduct is defined trivially to reproduce the usual tensor product representations of Lie algebra elements J^a and Lie group elements $\exp(x_a \mathrm{J}^a)$

$$\Delta(1) = 1 \otimes 1, \qquad \Delta(\mathrm{J}^a) = \mathrm{J}^a \otimes 1 + 1 \otimes \mathrm{J}^a. \tag{6.41}$$

Coproducts of polynomials X are defined by means of the compatibility relation. Note that the iterated coproduct defines the action of symmetry generators on a spin chain, e.g.[10]

$$\Delta^{L-1}(1) = 1, \qquad \Delta^{L-1}(\mathrm{J}^a) = \sum_{k=1}^{L} \mathrm{J}^a_k. \tag{6.42}$$

The unit and counit are defined as

$$\epsilon(1) = 1, \qquad \eta(1) = 1, \quad \eta(\mathrm{J}^a) = 0. \tag{6.43}$$

They implement the natural operations involving the unit element, and are hardly used in practice.

Finally, the antipode acts as

$$\Sigma(1) = 1, \qquad \Sigma(\mathrm{J}^a) = -\mathrm{J}^a. \tag{6.44}$$

Since the antipode acts as the negative on the Lie algebra, it acts as the inverse on the Lie group. In that sense, it should be viewed as a generalisation of the inversion operation. In plain enveloping algebras it acts as an involution, but in more general situations Σ^2 differs from the identity map.

Universal R-Matrix. The framework of Hopf algebras can be extended to incorporate the R-matrix of quantum integrable systems. We introduce the so-called *universal R-matrix* $\mathcal{R}$ which is an invertible algebraic element

$$\mathcal{R} \in \mathrm{A} \otimes \mathrm{A}. \tag{6.45}$$

The R-matrices which we have encountered so far should be viewed as representations $(\rho_1 \otimes \rho_2)(\mathcal{R})$ of the universal R-matrix.

The universal R-matrix relates the coproduct with the opposite coproduct

$$\mathcal{R}\Delta(X) = \tilde{\Delta}(X)\mathcal{R}. \tag{6.46}$$

[10] An iterated coproduct acts on any one of the intermediate tensor factors. The result does not depend on the choice of tensor factors because the coproduct is coassociative.

In other words, even though the coproduct is not strictly cocommutative, it is cocommutative up to conjugation by $\mathcal{R}$. This property is called *quasi-cocommutativity*. On the level of representations, the relation tells us that the tensor product of two representations $\rho_1 \otimes \rho_2$ is equivalent to the opposite one $\rho_2 \otimes \rho_1$.

The second important property called *quasi-triangularity* is

$$\Delta_1(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \qquad \Delta_2(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}. \tag{6.47}$$

Among others useful features, it implies the Yang–Baxter equation

$$\mathcal{R}_{12}(\mathcal{R}_{13}\mathcal{R}_{23}) = \mathcal{R}_{23}(\mathcal{R}_{13}\mathcal{R}_{12}). \tag{6.48}$$

This means that every R-matrix derived as a representation of the universal R-matrix satisfies the Yang–Baxter equation.

6.4 Yangian Algebra

We will now be more concrete about an algebra which is relevant to the Heisenberg XXX spin chain and generalisations, the so-called *Yangian algebra*. A noteworthy deformation is the *quantum affine algebra* which is relevant to XXZ-like spin chains.

Algebra. The Yangian algebra $\mathrm{Y}(\mathfrak{g})$ of a finite-dimensional simple Lie algebra $\mathfrak{g}$ is the algebra of polynomials in the elements J^a and $\widehat{\mathrm{J}}^a$ with $a = 1, \ldots, \dim \mathfrak{g}$. The elements J^a and $\widehat{\mathrm{J}}^a$ are called level-0 and level-1 generators, respectively.

The following identifications of polynomials apply:

$$[\mathrm{J}^a, \mathrm{J}^b] = f^{ab}_c \mathrm{J}^c. \tag{6.49}$$

In other words, the J^a generate the Lie algebra $\mathfrak{g}$. Furthermore,

$$[\mathrm{J}^a, \widehat{\mathrm{J}}^b] = f^{ab}_c \widehat{\mathrm{J}}^c. \tag{6.50}$$

In other words, the $\widehat{\mathrm{J}}^a$ transform in the adjoint representation of $\mathfrak{g}$. Finally, the so-called *Serre relation* must hold

$$\big[[\mathrm{J}^a, \widehat{\mathrm{J}}^b], \widehat{\mathrm{J}}^c\big] + 2 \text{ cyclic} = \tfrac{1}{6}\hbar^2 f^{ag}_d f^{bh}_e f^{ci}_f f_{ghi} \{\mathrm{J}^d, \mathrm{J}^e, \mathrm{J}^f\}. \tag{6.51}$$

The term on the r.h.s. is the totally symmetric product of three terms.

When the generators J^a and $\widehat{\mathrm{J}}^a$ are identified with the generators J^a_0 and J^a_1 of the half loop algebra $\mathfrak{g}[u]$, the Serre relation is a deformation of the Jacobi identity for J^a, $\widehat{\mathrm{J}}^b$, $\widehat{\mathrm{J}}^c$. Without the deformation term on the r.h.s., the Jacobi identity makes sure that iterated commutators of the generators J^a_1 yield the higher-level generators J^a_n and nothing else. The Yangian algebra $\mathrm{Y}(\mathfrak{g})$ therefore is a

deformation of the enveloping algebra $\mathrm{U}(\mathfrak{g}[u])$. The constant $\hbar$ is the parameter of the deformation.[11]

The Yangian is a Hopf algebra. The coproduct for the Yangian generators is defined as

$$\begin{aligned}\Delta(1) &= 1 \otimes 1,\\ \Delta(\mathrm{J}^a) &= \mathrm{J}^a \otimes 1 + 1 \otimes \mathrm{J}^a,\\ \Delta(\widehat{\mathrm{J}}^a) &= \widehat{\mathrm{J}}^a \otimes 1 + 1 \otimes \widehat{\mathrm{J}}^a + \hbar f^a_{bc} \mathrm{J}^b \otimes \mathrm{J}^c.\end{aligned} \tag{6.52}$$

The latter term is a deformation of the trivial coalgebra structure on the enveloping algebra. It is precisely compatible with the deformation of the algebra by means of the Serre relation. Furthermore, the antipode reads

$$\Sigma(\mathrm{J}^a) = -\mathrm{J}^a, \qquad \Sigma(\widehat{\mathrm{J}}^a) = -\widehat{\mathrm{J}}^a + \tfrac{1}{2}\hbar f^a_{bc} f^{bc}_d \mathrm{J}^d. \tag{6.53}$$

Evaluation Representations. The Yangian algebra is a deformation of the enveloping algebra of a half loop algebra. Therefore it is conceivable that evaluation representations lift to the Yangian algebra. For some representation ρ of $\mathfrak{g}$, there may exist a one-parameter family of representations ρ_u of $\mathrm{Y}(\mathfrak{g})$:

$$\rho_u(1) = 1, \qquad \rho_u(\mathrm{J}^a) = \rho(\mathrm{J}^a), \qquad \rho_u(\widehat{\mathrm{J}}^a) = u\,\rho(\mathrm{J}^a). \tag{6.54}$$

Note that the deformation can invalidate evaluation representations. In particular, the r.h.s. of the Serre relation must be zero for a valid representation. Interestingly this condition is an identity formulated in terms of ρ of $\mathfrak{g}$ alone. For $\mathfrak{su}(N)$ this poses no restrictions, but for example for the adjoint representations of $\mathfrak{so}(N)$ or $\mathfrak{e}_8$ the term is not zero. In the latter case, the sum of an adjoint and singlet representation can form a proper Yangian representation.

Spin Chains. To define the Yangian action on a homogeneous spin chain we pick the evaluation representation ρ_0 with homogeneous evaluation parameter $u = 0$ for every site

$$\rho_0(1) = 1, \qquad \rho_0(\mathrm{J}^a) = \rho(\mathrm{J}^a), \qquad \rho_0(\widehat{\mathrm{J}}^a) = 0. \tag{6.55}$$

The tensor product representation on the spin chain therefore reads

$$\rho_{\mathrm{ch}} = (\rho_0 \otimes \ldots \otimes \rho_0) \circ \Delta^{L-1}. \tag{6.56}$$

For the level-0 and level-1 generators the iterated coproduct amounts to

$$\begin{aligned}\Delta^{L-1}(\mathrm{J}^a) &= \sum_{k=1}^{L} \mathrm{J}^a_k.\\ \Delta^{L-1}(\widehat{\mathrm{J}}^a) &= \sum_{k=1}^{L} \widehat{\mathrm{J}}^a_k + \hbar f^a_{bc} \sum_{k<l=1}^{L} \mathrm{J}^b_k \mathrm{J}^c_l.\end{aligned} \tag{6.57}$$

[11] In fact, Yangian algebras with different parameter $\hbar \neq 0$ are equivalent. We are free to fix the parameter to any value. A conventional choice if $\pm i/2$.

The representations therefore read

$$\rho_{\rm ch}(\mathrm{J}^a) = \sum_{k=1}^{L} \rho_k(\mathrm{J}^a), \qquad \rho_{\rm ch}(\widehat{\mathrm{J}}^a) = \hbar f^a_{bc} \sum_{k<l=1}^{L} \rho_k(\mathrm{J}^b)\rho_l(\mathrm{J}^c). \tag{6.58}$$

Let us discuss how the spin chain Hamiltonian interacts with the Yangian algebra

$$\mathcal{H} = \sum\nolimits_k \mathcal{H}_{k,k+1}. \tag{6.59}$$

Our Hamiltonian was constructed such that it is manifestly symmetric under some Lie algebra $\mathfrak{g}$, e.g. $\mathfrak{su}(N)$

$$\big[\rho_{\rm ch}(\mathrm{J}^a), \mathcal{H}\big] = 0. \tag{6.60}$$

This follows immediately from the action on the Hamiltonian kernel $\mathcal{H}_{k,k+1}$

$$\big[(\rho_0 \otimes \rho_0) \circ \Delta(\mathrm{J}^a), \mathcal{H}_{12}\big] = 0. \tag{6.61}$$

The situation is different for the level-one generators where one finds

$$\big[(\rho_0 \otimes \rho_0) \circ \Delta(\widehat{\mathrm{J}}^a), \mathcal{H}_{12}\big] = \mathcal{X}^a_2 - \mathcal{X}^a_1 \tag{6.62}$$

with some operator $\mathcal{X}^a_k$ acting on a single site. The action on the complete Hamiltonian turns out to be a telescoping sum

$$\big[\rho_{\rm ch}(\widehat{\mathrm{J}}^a), \mathcal{H}\big] = \sum\nolimits_k \big(\mathcal{X}^a_{k+1} - \mathcal{X}^a_k\big). \tag{6.63}$$

Now we have to pay attention to boundary conditions. For a closed chain with Hamiltonian

$$\mathcal{H} = \sum_{k=1}^{L} \mathcal{H}_{k,k+1} \tag{6.64}$$

one finds

$$\big[\rho_{\rm ch}(\widehat{\mathrm{J}}^a), \mathcal{H}\big] = \mathcal{X}_L - \mathcal{X}_1 + \sum_{k=1}^{L-1} \big(\mathcal{X}^a_{k+1} - \mathcal{X}^a_k\big) = 2\mathcal{X}_L - 2\mathcal{X}_1. \tag{6.65}$$

Therefore Yangian *symmetry* is broken by periodic boundary conditions. The spectrum of the spin chain Hamiltonian does not organise itself according to representations of the Yangian algebra.[12] Nevertheless, the Yangian is a useful algebra for the construction of eigenstates as we shall see. It also makes sense to consider it as a symmetry of the bulk Hamiltonian up to boundary terms.

[12]Since the Yangian algebra is very large, its representations are typically large, too. If the Yangian was a symmetry, the degeneracies of eigenvalues would be very pronounced, up to the point that all eigenvalues are degenerate.

Magnon States. In the action of the Yangian on magnon states one can nicely observe the relationship between the momentum p, rapidity u and evaluation representations ρ_u.

For a spin chain with $\mathfrak{su}(N)$ fundamental spins, the residual symmetry of the magnon picture is $\mathfrak{u}(N-1)$. For the Heisenberg XXX chain we have the z-components of J^a and $\widehat{\mathrm{J}}^a$ at our disposal[13]

$$\mathrm{J}^z_{\mathrm{reg}} = \tfrac{1}{2}\sum_k(\sigma^z_k + 1), \qquad \widehat{\mathrm{J}}^z = \hbar\sum_{k<l}\big(\sigma^-_k\sigma^+_l - \sigma^+_k\sigma^-_l\big). \tag{6.66}$$

Here the level-zero generator was regularised such that it can act on an infinite spin chain. Its action on the vacuum is normalised to zero, and it measures the number of flipped spins, i.e. the magnon number

$$\mathrm{J}^z_{\mathrm{reg}}|p_1,\dots,p_M\rangle = M|p_1,\dots,p_M\rangle. \tag{6.67}$$

For the level-one generator acting on a single magnon we obtain

$$\widehat{\mathrm{J}}^z|p\rangle = \hbar\sum_{k<l}\big(e^{ipk}|l\rangle - e^{ipl}|k\rangle\big). \tag{6.68}$$

Reorganising the sums and ignoring any boundary terms on the infinite chain we find

$$\widehat{\mathrm{J}}^z|p\rangle = \hbar\sum_{l=1}^{\infty}\big(e^{-ipl} - e^{ipl}\big)\sum_k e^{ipk}|k\rangle. \tag{6.69}$$

In other words, the eigenvalue of $\widehat{\mathrm{J}}^z$ equals

$$u = \hbar\sum_{l=1}^{\infty}\big(e^{-ipl} - e^{ipl}\big) = \frac{\hbar}{1-e^{-ip}} - \frac{\hbar}{1-e^{ip}} = -i\hbar\cot(\tfrac{1}{2}p) \tag{6.70}$$

times the eigenvalue 1 of $\mathrm{J}^z_{\mathrm{reg}}$. By setting $\hbar = i/2$ we recover the relationship $u = \frac{1}{2}\cot(\frac{1}{2}p)$ between momentum p and rapidity u. This implies that a single magnon state transforms in an evaluation representation of the residual Yangian algebra $\mathrm{Y}(\mathfrak{su}(N-1))$ with rapidity u as the evaluation parameter. One can convince oneself that many-magnon partial eigenstates transform in tensor product representations with individual evaluation parameters determined by their momenta.

R-Matrix. The S-matrix acts on two-magnon partial eigenstates and it interchanges the order of constituent magnons. This implies that the Yangian action on the spin chain acts differently on the ingoing and outgoing two-magnon states. Symmetry of the S-matrix or the analogous R-matrix means

$$\Delta(X)\mathcal{R} = \mathcal{R}\tilde{\Delta}(X) \quad \text{for any } X \in \mathrm{Y}(\mathfrak{g}). \tag{6.71}$$

[13] We implicitly assume that the generators are in some representation, here the spin-1/2 representation.

Here $\tilde{\Delta}$ is the opposite coproduct which takes into account that the ordering of magnons has flipped. Concretely, for $X = \mathrm{J}^a, \widehat{\mathrm{J}}^a$

$$\begin{aligned}
\Delta(\mathrm{J}^a) &= \mathrm{J}^a \otimes 1 + 1 \otimes \mathrm{J}^a, \\
\Delta(\widehat{\mathrm{J}}^a) &= u(\mathrm{J}^a \otimes 1) + v(1 \otimes \mathrm{J}^a) + \hbar f^a_{bc} \mathrm{J}^b \otimes \mathrm{J}^c, \\
\tilde{\Delta}(\widehat{\mathrm{J}}^a) &= u(\mathrm{J}^a \otimes 1) + v(1 \otimes \mathrm{J}^a) - \hbar f^a_{bc} \mathrm{J}^b \otimes \mathrm{J}^c.
\end{aligned} \tag{6.72}$$

Evidently, the coalgebra is not cocommutative, but the relation $\tilde{\Delta}(X) = \mathcal{R}^{-1}\Delta(X)\mathcal{R}$ implies that the opposite coproduct is equivalent to the ordinary coproduct. This feature is called *quasi-cocommutativity*.

The relation for $X = \mathrm{J}^a$ and fundamental spins of $\mathfrak{su}(N)$ implies that $\mathcal{R}$ must be of the form

$$\mathcal{R} = R_1 \mathcal{I} + R_2 \mathcal{P} \tag{6.73}$$

with two unconstrained functions $R_{1,2}$. This follows from the fact that $\mathcal{I}$ and $\mathcal{P}$ are the only $\mathfrak{su}(N)$ invariant operators. For the level-one generator $X = \widehat{\mathrm{J}}^a$ we obtain additional constraints implying $-2\hbar R_1 = (u - v)R_2$. Therefore the R-matrix must be proportional to

$$\mathcal{R} \sim (u - v)\mathcal{I} + 2\hbar \mathcal{P} \tag{6.74}$$

matching our earlier results for $\hbar = i/2$.

Classical Limit. Let us make a brief digression to the classical r-matrix which should clearly be related to the R-matrix. Here one takes a classical limit of the R-matrix where $u, v \to \infty$ and u/v remains finite. One finds the classical r-matrix as the leading correction term

$$\mathcal{R} \simeq 1 \otimes 1 + ir, \qquad r = \frac{\mathcal{P} - \mathcal{I}}{u - v}\,. \tag{6.75}$$

Moreover, the cobracket of the Lie algebra can be obtained as the leading anti-symmetric part of the coproduct.

Tensor Products. The R/S-matrix acts on the tensor product of two representations. Let us investigate the latter.

In $\mathfrak{su}(N)$ the tensor product of two fundamental representations decomposes according to

$$\begin{gathered}
\square \otimes \square = \square\square \oplus \begin{smallmatrix}\square\\\square\end{smallmatrix}, \\
\text{fund.} \otimes \text{fund.} = \text{sym.} \oplus \text{anti-sym.}, \\
(\tfrac{1}{2}) \otimes (\tfrac{1}{2}) = (1) \oplus (0) \quad \text{for } \mathfrak{su}(2).
\end{gathered} \tag{6.76}$$

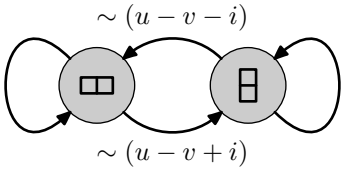
Since the Yangian algebra enhances the $\mathfrak{su}(N)$ Lie algebra, there is more: Consider three states in $\mathfrak{su}(2)$

$$\begin{aligned}
|0\rangle &:= |{\downarrow\downarrow}\rangle \in \square\square, \\
|\mathrm{s}\rangle &= |{\uparrow\downarrow}\rangle + |{\downarrow\uparrow}\rangle \in \square\square, \\
|\mathrm{a}\rangle &= |{\uparrow\downarrow}\rangle - |{\downarrow\uparrow}\rangle \in \begin{smallmatrix}\square\\\square\end{smallmatrix}.
\end{aligned} \tag{6.77}$$

Act with raising and lowering operators $\mathrm{J}^{\pm}$, $\widehat{\mathrm{J}}^{\pm}$ using $\rho := (\rho_u \otimes \rho_v) \circ \Delta$

$$\begin{aligned}
\Delta(\mathrm{J}^{+})|0\rangle &= |{\uparrow\downarrow}\rangle + |{\downarrow\uparrow}\rangle = |\mathrm{s}\rangle, \\
\Delta(\widehat{\mathrm{J}}^{+})|0\rangle &= u|{\uparrow\downarrow}\rangle + v|{\downarrow\uparrow}\rangle + \tfrac{i}{2}|{\uparrow\downarrow}\rangle - \tfrac{i}{2}|{\downarrow\uparrow}\rangle \\
&= \tfrac{1}{2}(u+v)|\mathrm{s}\rangle + \tfrac{1}{2}(v-u-i)|\mathrm{a}\rangle, \\
\Delta(\mathrm{J}^{-})|a\rangle &= 0, \\
\Delta(\widehat{\mathrm{J}}^{-})|a\rangle &= (v-u+i)|0\rangle.
\end{aligned} \tag{6.78}$$

These relations among others can be summarised in the following diagram:



(6.79)

The representation $\rho = \rho_u \otimes \rho_v$ has an unconventional structure from the point of view of finite-dimensional simple Lie algebras. The different cases are summarised in the following table:

decomposable	reducible indecomposable	irreducible
never in $\mathrm{Y}[\mathfrak{g}]$ (only in $\mathfrak{g}$)	for $u-v=\pm i$	$u-v \neq \pm i$ ($\mathcal{R}$ almost always fixed)

(6.80)

Fusion. Note that the configuration $u - v = \pm i$ has appeared in several contexts:

- tensor product representations,
- poles and zeros of R/S-matrices,
- bound states of magnons,
- numerator and denominator of the Bethe equations.

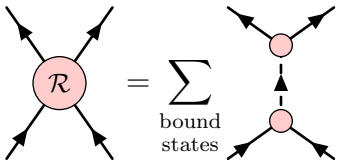
These occurrences are all related: Given the structure of the Yangian action at $u - v = \pm i$, namely

$$\Delta(X) \sim \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \quad \text{but} \quad \tilde{\Delta}(X) \sim \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}, \tag{6.81}$$

one can convince oneself that $\mathcal{R}\Delta(X) = \tilde{\Delta}(X)\mathcal{R}$ implies that $\mathcal{R}$ cannot have maximum rank at $u - v = \pm i$. Hence there must be zeros.

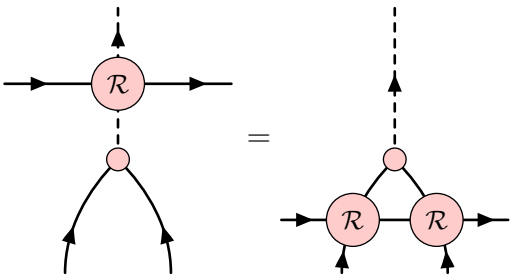
For an S-matrix one would like to implement the relation $S_{21} = S_{12}^{-1}$ strictly. This implies the existence of poles at these locations to compensate for the zeros in S_{21}.

Poles in the S-matrix indicate the presence of bound states. Two particles in a suitable configuration can form a bound state whose propagator manifests as a pole



(6.82)

The S-matrix for scattering with bound states can be obtained as a product of scattering processes with the constituents



(6.83)

For the R-matrix there is an analogous relation called *quasi-triangularity*

$$\Delta_1(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \qquad \Delta_2(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}. \tag{6.84}$$

These relation in fact imply the Yang–Baxter equation. They can also be used to determine the R-matrix for higher representations.

The poles of the S-matrix can be used to recursively construct the spectrum of all bound state particles and their symmetries. This is called the bootstrap:

- Start with the S-matrix of some particles.
- Find all poles of all available S-matrices.
- Compute the S-matrices for these bound states from quasi-triangularity.
- Repeat the previous steps with the enlarged set of bound states.
- Stop when all poles of all S-matrices have been accounted for.

Note that $\mathcal{R} = (u-v)\mathcal{I} + i\mathcal{P}$ has zeros at $u - v = \pm i$, i.e. $\mathcal{R} = \pm i(\mathcal{I} \pm \mathcal{P})$ becomes a projector. Therefore the R-matrix is sometimes (ab)used to project to sub-representations. For example, the first R-matrix in the following combination projects the space 12 to a symmetric combination

$$\mathcal{R}_{12}(u + \tfrac{i}{2}, u - \tfrac{i}{2})\mathcal{R}_{13}(u + \tfrac{i}{2}, v)\mathcal{R}_{23}(u + \tfrac{i}{2}, v). \tag{6.85}$$

7 AdS/CFT Integrability

Topics in this chapter:

- CFT, local operators, anomalous dimensions,
- SU(2) sector, spin chain picture, Hamiltonian
- SL(2) = SU(1, 1) sector, Bethe equations, PSU(2, 2|4) spin chain
- higher-loop corrections, long-range interactions, long-range Bethe equations
- wrapping terms, Lüscher Corrections, very high loop anomalous dimensions, multiple zeta values
- mirror theory, TBA, T/Y-system, NLIE, quantum curve thermodynamic limit
- three-point functions, partial scalar products, determinant formulas
- scattering amplitudes, null polygonal Wilson loops, worldsheet area, T-duality, weak-weak duality, dual conformal symmetry, Yangian symmetry, problem of divergences

For further details, please refer to reviews on this subject.