Exercise 1. *Canonical quantization of phonons*

We consider a chain of N atoms of mass m and coordinates x_n, with $n = 1, \ldots, N$. The atoms interact through a potential $V(x_n) = V(x_1, \ldots, x_N)$ that can be written in the *harmonic approximation* as

$$V(u_1, \ldots, u_N) = \frac{\lambda}{2} \sum_{n=1}^{N} (u_{n+1} - u_n)^2 + \frac{1}{2} m\Omega^2 \sum_{n=1}^{N} (u_n)^2, \tag{1}$$

where $u_n = x_n - \bar{x}_n$, $|u_n| \ll \bar{x}_n$ measures the (small) deviation from the equilibrium position of each atom $\bar{x}_n = na$, a being the lattice constant. In (1) λ is the elastic constant of the chain and the Ω term constrains each atom at its equilibrium position. The kinetic energy of the atoms is readily written as:

$$T(\dot{u}_1, \ldots, \dot{u}_N) = \frac{1}{2} m \sum_{n=1}^{N} (\dot{u_n})^2. \tag{2}$$

a) After writing down the classical Lagrangian of the system and the corresponding equations of motion (Euler-Lagrange equations), solve for the normal modes by imposing periodic boundary conditions (PBC, $u_n = u_{n+N}$). What are the symmetries of the system for $\Omega = 0$? Comment on the resulting spectrum in the two cases $\Omega \neq 0$ and $\Omega = 0$. In the latter case, how does the spectrum look like in the *long-wavelength* regime?

Hint. Solve the E-L equations with an exponential ansatz $u_n \propto e^{i(kna - \omega t)}$ and impose PBC to obtain the normal modes $\omega_l = \omega(k_l)$. The general solution will look like:

$$u_n(t) = \sum_{l=1}^{N} [A_l e^{i(k_l na - \omega_l t)} + c.c.], \tag{3}$$

where the A_l are fixed by the initial conditions. The long-wavelength regime is characterized by $k_l \ll 1$.

b) Identify the conjugated momenta $\pi_n(t)$ and write down the Hamiltonian for the system through a Legendre transformation.

c) We set $m = 1$ and introduce the dimensionless operators $\hat{a}_l = \sqrt{N}\sqrt{\frac{2\omega_l}{\hbar}}\hat{A}_l$, satisfying $[\hat{a}_l, \hat{a}^\dagger_{l'}] = \delta_{ll'}$. Use them to elevate $u_n(t), \pi_n(t)$ to quantum operators $\hat{u}_n(t), \hat{\pi}_n(t)$. Prove the equal-time canonical commutation relation $[\hat{u}_n(t), \hat{\pi}_{n'}(t)] = i\hbar\delta_{nn'}$.

d) Write the Hamiltonian in (c) in terms of the new operators $\hat{a}_l, \hat{a}^\dagger_l$. Write a general eigenstate. What is the ground state? What defines an excited state?

Exercise 2. *Planets as blackbodies?*

The Stefan-Boltzmann law states that the emission power per unit surface area of a blackbody reads

$$P_{em} = \sigma T^4 \quad \text{with} \quad \sigma = \frac{\pi^2 k_B^4}{60\hbar^3 c^2} \approx 5.6704 \cdot 10^{-8}\ \text{Js}^{-1}\text{m}^{-2}\text{K}^{-4}. \tag{4}$$

a) Making use of the Stefan-Boltzmann law, estimate the temperature of the Earth, Mars and Venus as if they were blackbodies.

   Hint. The energy emitted and absorbed has to balance.

b) The correct results for the average temperatures are 288 K for the Earth, 218 K for Mars and 735 K for Venus. How do they compare with the estimates in (a)? What could be the reasons of the discrepancies?