Exercise 1. ***Pair correlation functions for fermions at finite temperature***

In this exercise we want to study the correlation functions for a system of free independent fermions at finite temperature, especially in the high temperature limit.

(a) Evaluate the thermal average $\langle \hat{O} \rangle = \frac{\mathrm{tr}\left\{e^{-\beta H'}\hat{O}\right\}}{\mathrm{tr}\, e^{-\beta H'}}$ for $\hat{O} = \hat{n}_{\vec{k}}$ and $\hat{O} = \hat{n}_{\vec{k}}\hat{n}_{\vec{q}}$ at $T = 0$ and at $T > 0$, where $H' = H - \mu\hat{N}$ and $\beta = \frac{1}{k_B T}$.

(b) Show that the one-particle correlation function is

$$\frac{n}{2} g_s(\vec{R}) = \iiint \frac{d^3k}{(2\pi)^3} n_{\vec{k}} e^{-i\vec{k}\cdot\vec{R}} \,, \tag{1}$$

where $n_{\vec{k}}$ is the Fermi-Dirac distribution.

(c) Show that in the high temperature limit

$$g_s(\vec{R}) \approx e^{-\frac{\pi\vec{R}^2}{\lambda^2}} \,, \tag{2}$$

where $\lambda = \sqrt{\frac{h^2}{2\pi m k_B T}}$ is the thermal wavelength. Compare this result with the one you know for $T = 0$.

Hint. $\int_{-\infty}^{\infty} e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}$ $\forall a \in \mathbb{R}_+, b \in \mathbb{C}$.

(d) Show that in the high temperature limit

$$g(\vec{R}) = \frac{g_{\uparrow\uparrow}(\vec{R}) + g_{\uparrow\downarrow}(\vec{R})}{2} \approx 1 - \frac{e^{-\frac{2\pi\vec{R}^2}{\lambda^2}}}{2} \,. \tag{3}$$

Compare this result with the one you know for bosons.

(e) How does the density depletion change? It is defined as $n \iiint d^3r \,(g(\vec{r}) - 1)$.

Exercise 2. ***Single-particle correlation function for bosons***

Consider a homogeneous gas of free independent spin-0 bosons at $T > T_c$. The single-particle correlation function is given by

$$g(\vec{R}) = \iiint \frac{d^3k}{(2\pi)^3} n_{\vec{k}} e^{-i\vec{k}\cdot\vec{R}} \,, \tag{4}$$

where $\epsilon_{\vec{k}} = \frac{\hbar^2\vec{k}^2}{2m}$ and $n_{\vec{k}} = \frac{1}{e^{\beta(\epsilon_{\vec{k}}-\mu)}-1}$ (Section 3.7.2 from the Lecture Notes).

(a) Show that in the $\vec{R} \to 0$ limit

$$g(\vec{R}) \approx n \left(1 - \frac{\vec{R}^2}{6} \langle \vec{k}^2 \rangle \right) , \tag{5}$$

where n is the particle density.

(b) Study $\langle \vec{k}^2 \rangle$ in the low and high temperature limits and derive the correlation function $g(\vec{R})$ in these limits. Express the result in terms of the thermal wave length $\lambda = \sqrt{\frac{h^2}{2\pi m k_B T}}$.

Hint. $\int_0^\infty dx\, x^{2n} e^{-ax^2} = \sqrt{\frac{\pi}{a}} \frac{1 \cdot 3 \cdots (2n-1)}{a^n 2^{n+1}} \quad \forall a \in \mathbb{R}_+, n \in \mathbb{N}$.

Hint. $\zeta(x)\Gamma(x) = \int_0^\infty du \frac{u^{x-1}}{e^u - 1} \quad \forall x > 1$.

(c) How would you modify the previous result for the correlation function to describe the Bose-Einstein condensate regime, too?