Exercise 1. ***Ising Model: Variational Approach vs. Mean Field.***

Consider an Ising lattice in d dimensions, where each of the N spins takes values $s_i = \pm s$ and has z nearest neighbors. In the presence of an external magnetic field H, the Hamiltonian is

$$\mathcal{H} = -J \sum_{\langle s_i, s_j \rangle} s_i \, s_j - \sum s_i H \; . \tag{1}$$

Let us also define the average magnetization of a spin as $s\,\tilde{m} = M/N = s\,(N_+ - N_-)\,/N$.

(a) Calculate the configurational entropy of the system (or remember that we already did that in Exercise Sheet 1). Determine the internal energy by taking the average value of the Hamiltonian, by making the approximation that each of the spins s_i independently takes the value ± 1 with probability $w_\pm = \frac{1}{2}\,(1 \pm \tilde{m})$. Write your answers in terms of $\tilde{m}$.

Hint. If s_i and s_j are independent, their expectation factorizes: $\langle s_i s_j \rangle = \langle s_i \rangle \langle s_j \rangle$.

(b) Determine the free energy of the system using the formula $F = U - TS$. Which variational principle determines the magnetization of the system? Derive the corresponding equation. Compare with (5.21) in the lecture notes, which was obtained with the mean field approximation.

Exercise 2. ***Ising Model: Infinite-Range Forces and Mean Field.***

Consider an Ising model where now *all* spins interact between each other with the same strength $J = 1/N$ (long-range forces). The Hamiltonian is given by

$$\mathcal{H} = -\frac{1}{2N} \sum_{i,\,k} s_i s_k - H \sum s_i \; . \tag{2}$$

The coupling constant is rescaled by N so that the total energy remains finite; also the factor one-half compensates the fact that in the sum, each index i and k ranges independently from 1 to N, and thus we counted each bond twice.

In this exercise, we'll show that the mean-field approach for this model is exact (at least for $N \to \infty$).

(a) In order to calculate the partition function for this model, we will introduce a little mathematical trick. Show that the Boltzmann factor which appears in the partition function can be written as

$$\mathrm{e}^{-\beta\mathcal{H}} = \sqrt{\frac{N\beta}{2\pi}} \int_{-\infty}^{\infty} d\lambda \exp\left(-\frac{N\beta\lambda^2}{2} + \sum_i \beta\,(\lambda + H)\,s_i \right) \; . \tag{3}$$

This is a particular case of the Gaussian transform method which will be seen in the lecture.

Hint. You should know the Gaussian integral by heart by now, but just in case: $\int dx\, \mathrm{e}^{-ax^2} = \sqrt{\frac{\pi}{a}}$.

(b) Show that the partition function can be written as

$$Z = \sqrt{\frac{N\beta}{2\pi}} \int d\lambda \, \mathrm{e}^{-N\beta A(\lambda)} \ , \qquad A(\lambda) = \frac{\lambda^2}{2} - \frac{1}{\beta} \ln\left(2\cosh\left[\beta(\lambda + H)\right]\right) \tag{4}$$

In order to determine the partition function, we will use the steepest descent method (a.k.a. Laplace method or saddle point approximation): the integral of the exponential is dominated by the maximum of the function in the exponential. Technically this is done by expanding the function in the exponent to second order at its maximum, and neglecting further orders.

(c) Determine the equation that λ should satisfy in order for it to be the maximum of the argument of the exponential.

Show that the partition function can be written (for large N) as

$$Z \approx \mathrm{e}^{-N\beta f} \ ; \qquad f = A(\lambda_0) + \frac{1}{2N\beta} \ln A''(\lambda_0) \approx A(\lambda_0) \ , \tag{5}$$

where f is the free energy per spin and λ_0 is the minimum of the function $A(\lambda)$.

Show that λ_0 is precisely the average magnetization of a spin, $\lambda_0 = \langle s_i \rangle =: m$. Deduce that your result coincides with the magnetization that you would get via mean field theory.

Hint. The average magnetization per spin can be obtained via the free energy per spin, $m = -\dfrac{\partial f}{\partial H}$.