Exercise 1. ***The Bogolyubov transformation.***

We consider a gas of weakly interacting bosonic particles at low temperatures. In this limit, the corresponding Hamiltonian can be approximated by

$$\mathcal{H} = \tfrac{1}{2}U\Omega n_0^2 - \mu\Omega n_0 + \tfrac{1}{2}\sum_{\boldsymbol{k}\neq 0}\left\{(\epsilon_{\boldsymbol{k}} - \mu + 2Un_0)\left(\hat{a}_{\boldsymbol{k}}^\dagger \hat{a}_{\boldsymbol{k}} + \hat{a}_{-\boldsymbol{k}}^\dagger \hat{a}_{-\boldsymbol{k}}\right) + Un_0\left(\hat{a}_{\boldsymbol{k}}^\dagger \hat{a}_{-\boldsymbol{k}}^\dagger + \hat{a}_{\boldsymbol{k}}\hat{a}_{-\boldsymbol{k}}\right)\right\}, \tag{1}$$

where $\epsilon_{\boldsymbol{k}}$ is the free dispersion,

$$\epsilon_{\boldsymbol{k}} = \frac{\hbar^2 k^2}{2m}. \tag{2}$$

(a) Introduce quasiparticle annihilation and creation operators $\hat{\gamma}_{\boldsymbol{k}}$ and $\hat{\gamma}_{\boldsymbol{k}}^\dagger$ which are defined by the relation

$$\hat{a}_{\boldsymbol{k}} = u_{\boldsymbol{k}}\hat{\gamma}_{\boldsymbol{k}} - v_{\boldsymbol{k}}\hat{\gamma}_{-\boldsymbol{k}}^\dagger \quad \text{and} \quad \hat{a}_{-\boldsymbol{k}} = u_{\boldsymbol{k}}\hat{\gamma}_{-\boldsymbol{k}} - v_{\boldsymbol{k}}\hat{\gamma}_{\boldsymbol{k}}^\dagger. \tag{3}$$

What is the condition for $u_{\boldsymbol{k}}$ and $v_{\boldsymbol{k}}$ in order to obtain bosonic commutation relations for these operators?

(b) For real-valued $u_{\boldsymbol{k}}$ and $v_{\boldsymbol{k}}$ you can write the transformation coefficients as

$$u_{\boldsymbol{k}} = \frac{1}{\sqrt{1-\chi_{\boldsymbol{k}}^2}} \quad \text{and} \quad v_{\boldsymbol{k}} = \frac{\chi_{\boldsymbol{k}}}{\sqrt{1-\chi_{\boldsymbol{k}}^2}}. \tag{4}$$

Determine the function $\chi_{\boldsymbol{k}}$ such that the Hamiltonian is diagonal in the quasiparticle operators,

$$\mathcal{H} = E_0 - \mu\Omega n_0 + \tfrac{1}{2}\sum_{\boldsymbol{k}\neq 0} E_{\boldsymbol{k}}\left(\hat{\gamma}_{\boldsymbol{k}}^\dagger\hat{\gamma}_{\boldsymbol{k}} + \hat{\gamma}_{-\boldsymbol{k}}^\dagger\hat{\gamma}_{-\boldsymbol{k}}\right). \tag{5}$$

(c) Find the quasiparticle dispersion $E_{\boldsymbol{k}}$. Fix the chemical potential μ in such a way that the energy spectrum is linear for $\boldsymbol{k} \to 0$. Approximate the dispersion for small ($\boldsymbol{k} \to 0$) and large ($\epsilon_{\boldsymbol{k}} \gg Un_0$) momenta and calculate the sound velocity for $k \to 0$.

Exercise 2. ***Temperature dependence of the superfluid fraction.***

In the lecture we calculated the number of condensed (superfluid) particles at zero temperature [Eq. (6.31)]. In this exercise we want to determine the temperature dependence of this fraction in the limit $T \to 0$.

(a) Calculate the expectation value of the density of particles with momentum $\boldsymbol{k}$,

$$n_{\boldsymbol{k}} := \frac{1}{\Omega}\left\langle \hat{a}_{\boldsymbol{k}}^\dagger \hat{a}_{\boldsymbol{k}}\right\rangle. \tag{6}$$

Hint. Use the fact that the Bogolyubov quasiparticles defined in Eq. (3) follow a Bose-Einstein distribution.

(b) Approximate the temperature dependence of this density,

$$\delta n_{\boldsymbol{k}}(T) := n_{\boldsymbol{k}}(T) - n_{\boldsymbol{k}}(T=0), \tag{7}$$

in the limit $T \to 0$.

(c) Calculate the temperature dependence of the density of condensed particles,

$$\delta n_0 = -\sum_{\boldsymbol{k}} \delta n_{\boldsymbol{k}}, \tag{8}$$

in the same limit. What happens in a two-dimensional system?

Hint. Keep only the terms of lowest order in T.

(d) Calculate the expectation value $\left\langle \hat{a}_{\boldsymbol{k}}^\dagger \hat{a}_{-\boldsymbol{k}}^\dagger\right\rangle$. What is the physical interpretation of this quantity?