Sheet I

Return by 26.9.2013

Question 1 [*Orbit-Stabiliser Theorem*]: Let X be a set and G be a group. We say that G has an action on X if, for each element $g \in G$, we have a map

$$g : X \to X \ , \qquad x \mapsto g \cdot x$$

such that

$$e \cdot x = x \ , \quad (g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x) \qquad \forall g_1, g_2 \in G \ , \ \forall x \in X \ , \tag{1}$$

where e is the identity element of the group G. Thus G has an action on X if G can be considered as a group of transformations acting on X.

Given a group action on X, we define the *stabiliser* of an element $x \in X$ as the subset of transformations that map x onto itself,

$$\text{Stab}(x) := \Big\{ g \in G \,|\, g \cdot x = x \Big\} \ . \tag{2}$$

The *orbit* of an element $x \in X$ under the action of G is the subset of X whose elements can be obtained by acting on x with some element of G,

$$Gx := \Big\{ y \in X \,|\, y = g \cdot x \text{ for some } g \in G \Big\} \ . \tag{3}$$

The orbit-stabiliser theorem states that the order $|G|$ of G can be calculated as the product of the order of the stabiliser of x times the cardinality of the orbit Gx

$$\text{card}(Gx) \cdot |\text{Stab}(x)| = |G| \ . \tag{4}$$

This is true for any $x \in X$.

The aim of this question is to verify this theorem in a simple (nontrivial) case, where G is the symmetry group of the cube (named O). Furthermore, one can relatively easily prove the statement abstractly.

(a) Enumerate the elements of O in terms of transformations of the cube. Do not include inversions (therefore consider only proper rotations). [*Hint:* $|O| = 24$.]

(b) Verify the orbit-stabiliser theorem by considering the action of O on

- the set F of faces of a cube;
- the set V of vertices of a cube.

*(c) Prove the orbit-stabiliser theorem in the general case.

Hint: fix an element $x \in X$, then

(i) show that the relation $g \sim h$ if $g \cdot x = h \cdot x$ is an equivalence relation;

(ii) show that the number of elements of G in each equivalence class is equal (and compute it!);

(iii) show that the number of equivalence classes into which G is partitioned via $\sim$ is equal to the cardinality of Gx.

Question 2 [*Dihedral group — Part I*]: The goal of this exercise is to gain familiarity with the dihedral group D_n and its irreducible representations. Recall that D_n is generated by two elements d and s satisfying the relations

$$d^n = s^2 = e\ , \qquad d^{-k}s = s\,d^k \quad \forall k \in \mathbb{Z}\ . \tag{5}$$

(a) The simplest dihedral group is D_3, the symmetry group of an equilateral triangle. Identify the elements d and s with symmetries of the triangle and convince yourself that the relations (5) are satisfied. Draw the multiplication table of D_3.

Recall that a representation ρ of a group G on a complex vector space V is a group homomorphism $\rho : G \to \mathrm{GL}(V)$, i.e. a map ρ such that $\rho(g_1 \cdot g_2) = \rho(g_1) \circ \rho(g_2)$.

(b) Suppose that S and D are $l \times l$ matrices satisfying

$$D^n = S^2 = \mathbf{1}_l, \qquad S\,D^k\,S = D^{-k} \quad \forall k \in \mathbb{Z}\ . \tag{6}$$

Show that $\rho(s) = S$ and $\rho(d) = D$ then defines an l-dimensional representation of D_n (acting by $l \times l$ matrices on an l-dimensional vector space).

(c) For the case of D_3 find three inequivalent irreducible representations of dimensions 1, 1 and 2. (We will see later that these are actually all representations of D_3.)

(d) Similarly, for the case of D_4 find five inequivalent irreducible representations of dimensions 1, 1, 1, 1, and 2. Again, this list will turn out to be complete.

*(e) For the case of D_5 find four inequivalent irreducible representations of dimensions 1, 1, 2, and 2. Once again, this list is actually complete.

Hints: A representation $\rho : G \to \mathrm{GL}(V)$ is called *irreducible* if there exists no proper non-zero subspace $W \subset V$ that is invariant under $\rho(a)$ for all $a \in G$. Moreover, two representations ρ_1 and ρ_2 of a group G on vector spaces V_1 and V_2 are said to be *equivalent* if there exists an isomorphism $T : V_1 \to V_2$ such that

$$T \circ \rho_1(a) = \rho_2(a) \circ T\ , \qquad \forall a \in G\ .$$