Sheet VI

Return by 31.10.2013

Question 1 [*Dimensions of irreducible representations of S_4*]:

(a) Recall from the lecture that we have seen three different ways of computing the dimensions of the irreducible representations of the symmetric groups: the one directly obtained from Frobenius theorem, the hook formula, and the enumeration of different Young tableaux of shape λ. Use all three formulae to compute the dimensions of the irreducible representations of S_4, showing that the answers agree with what you had already computed.

(b) Use Frobenius theorem to compute the characters of the different irreducible representations of S_4, and compare the result with what you derived before.

Question 2 [*Characters of S_n*]: Show that if g is a cycle of length n in S_n, then

$$\chi_\lambda(g) = \begin{cases} (-1)^s & \text{if } \lambda = \big(n-s, 1, \ldots, 1\big), \text{ where } 0 \leq s \leq (n-1) \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Question 3 [*Hook formula from Frobenius formula*]: The aim of this exercise is to deduce the hook length formula from the Frobenius formula. Recall that, given a partition $\lambda = \big(\lambda_1, \ldots, \lambda_k\big)$, the Frobenius dimension formula for S_n gives

$$\dim(V_\lambda) = \frac{n!}{l_1! \ldots l_k!} \prod_{i<j} \big(l_i - l_j\big) , \qquad \text{where} \qquad l_i = \lambda_i + k - i . \tag{2}$$

On the other hand, the hook formula reads

$$\dim(V_\lambda) = \frac{n!}{\prod_{(i,j)\in\lambda} h(i,j)} , \tag{3}$$

where $h(i,j)$ is the hook length of the box (i,j).

(a) Given an arbitrary Young diagram, identify the boxes whose hook length is equal to the l_i's of the same diagram.

(b) Prove that the Frobenius formula and the hook length formula give the same result for any diagram.

Hint: use induction and proceed as follows:

(i) first prove it for a diagram with a single column[1];

(ii) prove that, if the two are equal for a diagram, then they are also equal for the diagram where an additional box has been added on the bottom left;

(iii) finally, prove that if they are equal for a given diagram, they are also equal for the diagram where an additional column of the same length as the first column has been added on the left.

[1]Actually, it is enough to prove it for a diagram with a single box.