Sheet VII

Return by 14.11.2013

Question 1 [*The Lie algebra* $\mathfrak{so}(4)$]:

(a) Determine the generators and their commutators for the Lie algebra $\mathfrak{so}(4)$. Here $\mathfrak{so}(4)$ is the Lie algebra associated to the group SO(4) consisting of real orthogonal 4×4 matrices with determinant 1.

(b) Show that, as a real Lie algebra, $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, where the direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is defined to be the direct sum of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ as vector spaces with the additional requirement that $[\mathfrak{g}_1, \mathfrak{g}_2] = 0$.

Question 2 [*BCH formula*]:

The Baker-Campbell-Hausdorff (BCH) formula states that

$$\exp(\Omega_1) \cdot \exp(\Omega_2) = \exp(\Omega_1 \star \Omega_2) \ , \tag{1}$$

where $\Omega_1 \star \Omega_2$ is an element of the Lie algebra generated by Ω_1 and Ω_2 and equals

$$\Omega_1 \star \Omega_2 = \Omega_1 + \Omega_2 + \tfrac{1}{2}[\Omega_1, \Omega_2] + \tfrac{1}{12}[\Omega_1, [\Omega_1, \Omega_2]] + \tfrac{1}{12}[\Omega_2, [\Omega_2, \Omega_1]] + \cdots \ . \tag{2}$$

Prove the BCH formula to this order.

[*Hint:* Replace Ω_i by $t\Omega_i$ $(i = 1, 2)$ and expand both sides to the appropriate order in t.]

Question 3 [*Simple Lie groups and simple Lie algebras*]:

A simple Lie group is a connected non-abelian Lie group with no proper connected normal subgroups (a subgroup H of a group G is called *normal* if for all $h \in H$ and $g \in G$, $ghg^{-1} \in H$). We want to understand what this condition means in terms of the Lie algebra of the group. In order to do this, we will have to introduce some additional algebraic notions. In this exercise, we will always consider compact connected Lie groups, for which the exponential map is onto, i.e. each $g \in G$ can be written as $g = e^A$ for some $A \in \mathfrak{g}$.

(a) A *subalgebra* is a subset of an algebra which is closed under multiplication. In the case of a Lie algebra, this means that, given an algebra $\mathfrak{g}$, a subalgebra $\mathfrak{h}$ obeys

$$\mathfrak{h} \subseteq \mathfrak{g} \, : \, [\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}. \tag{3}$$

Show that the exponential map maps a subalgebra $\mathfrak{h}$ into a subgroup H of G.

[*Hint:* Use the BCH formula.]

(b) An *ideal* is a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ with the property that

$$[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}. \tag{4}$$

A proper ideal of $\mathfrak{g}$ is an ideal which is neither trivial nor all of $\mathfrak{g}$. Show that a Lie group is simple if and only if its Lie algebra contains no proper ideals.