Sheet XI

Return by 12.12.2013

Question 1 [*Representation theory of* $\mathfrak{sl}(5,\mathbb{C})$]: In this excercise we develop the representation theory of the complexification $\mathfrak{sl}(5,\mathbb{C})$ of the Lie algebra $\mathfrak{su}(5)$.

(a) Identify the Cartan subalgebra $\mathfrak{h}$, and define a suitable basis L_i, $i = 1,2,3,4,5$ subject to $\sum_j L_j \cong 0$ for the dual space $\mathfrak{h}^*$. Find the roots of the algebra and describe them in terms of elements of $\mathfrak{h}^*$.

(b) Identify subalgebras $\mathfrak{sl}(2,\mathbb{C})$ inside $\mathfrak{sl}(5,\mathbb{C})$, and deduce the structure of the possible weights of any finite-dimensional representation of $\mathfrak{sl}(5,\mathbb{C})$.

(c) Demonstrate that, as a finite group, the quotient of the weight lattice by the root lattice equals $\mathbb{Z}_5$.

(d) Choose a linear functional on the root space, and divide the roots into positive and negative roots. Then identify the possible highest weights of any finite dimensional representation. Show that these highest weights can be labelled by four non-negative integers. Identify these Dynkin labels for the fundamental (5-dimensional) representation of $\mathfrak{sl}(5,\mathbb{C})$, as well as for the conjugate of the fundamental representation.

Question 2 [SU(5) *Grand Unified Theory*]: According to the Standard Model of particle physics, all known particles fall into representations of the Lie group $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$. After the successful unification of the electromagnetic and the weak interactions into the theory of electroweak interactions, there have been various attempts to unify the remaining gauge interactions into a common simple gauge group. One of these *Grand Unified Theories* proposes to take SU(5) as the underlying gauge group, which is then broken down to $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ at a certain energy scale. This exercise is meant to guide you through the most basic ideas of this unification.

(a) Find an embedding of the Lie algebra $\mathfrak{su}(3)\oplus\mathfrak{su}(2)\oplus\mathfrak{u}(1)$ into $\mathfrak{su}(5)$. (Here $\mathfrak{u}(1)$ is the 1-dimensional abelian Lie algebra consisting of the set $i\mathbb{R}$ with vanishing Lie bracket.) Upon complexification, thus obtain a subalgebra $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{u}(1)_{\mathbb{C}}$ of $\mathfrak{sl}(5,\mathbb{C})$.

(b) Let V denote the fundamental representation of $\mathfrak{sl}(5,\mathbb{C})$. Decompose the representation $\Lambda V = \bigoplus_{i=0}^{5}\Lambda^i V$ into irreducible representations of $\mathfrak{g}_{\mathbb{C}}$. They form a generation of elementary *matter particles*. (To obtain all three generations, three copies of this representation are needed.)

First note that $\Lambda^0 V$ is just the trivial representation. Then analyse the decomposition of $\Lambda^1 V = V$ into irreducible representations of $\mathfrak{g}_{\mathbb{C}}$. Do the same analysis for $\Lambda^2 V$, using that $\Lambda^2(V_1\oplus V_2)\cong\Lambda^2 V_1\oplus(V_1\otimes V_2)\oplus\Lambda^2 V_2$.

In total, you should find the following matter (and antimatter) particles:

Name	$\mathfrak{su}(3)$	$\mathfrak{su}(2)$	$\mathfrak{u}(1)$
$\begin{pmatrix} e_R^+ \\ \overline{\nu}_R \end{pmatrix}$	$\mathbf{1}$	$\mathbf{2}^* = \mathbf{2}$	1
$\begin{pmatrix} u_L^r, u_L^g, u_L^b \\ d_L^r, d_L^g, d_L^b \end{pmatrix}$	$\mathbf{3}$	$\mathbf{2}$	$\frac{1}{3}$
ν_R	$\mathbf{1}$	$\mathbf{1}$	0
e_L^+	$\mathbf{1}$	$\mathbf{1}$	2
$(\overline{u}_L^{\bar{r}}, \overline{u}_L^{\bar{g}}, \overline{u}_L^{\bar{b}})$	$\mathbf{3}^*$	$\mathbf{1}$	$-\frac{4}{3}$
(d_R^r, d_R^g, d_R^b)	$\mathbf{3}$	$\mathbf{1}$	$-\frac{2}{3}$

For example, the first row consists of the right-handed antileptons, which transform in the trivial representation with respect to $\mathfrak{su}(3)$, in the two-dimensional (defining) representation of $\mathfrak{su}(2)$, while their eigenvalue with respect to $\mathfrak{u}(1)$ is 1. Physically speaking, this means they are colourless (or black), have weak isospin $T_3 = \pm\frac{1}{2}$ and weak hypercharge $Y = 1$. Their electrical charge can then be computed according to $Q = T_3 + Y/2$, so $Q = 0$ for $\overline{\nu}_R$ and $Q = 1$ for e_R^+, as it should be. The $\mathbf{3}^*$ in the fifth row denotes the conjugate of the fundamental representation of $\mathfrak{su}(3)$, and similarly for the $\mathbf{2}^*$ in the first row.

Finally, notice that $\Lambda^3 V$, $\Lambda^4 V$ and $\Lambda^5 V$ are just the representations conjugate to $\Lambda^2 V$, $\Lambda^1 V$ and $\Lambda^0 V$, respectively. Physically speaking, this means that they describe the corresponding antiparticles (i.e. all quantum numbers are multiplied by -1 and the chirality is opposite).

(c) Decompose the adjoint representation of $\mathfrak{sl}(5, \mathbb{C})$ into irreducible representations of $\mathfrak{g}_{\mathbb{C}}$. These irreducible representations are called *gauge bosons*. You should obtain an 8-dimensional representation (the gluons), a 3- and a 1-dimensional one (which, after electroweak symmetry breaking, become the W and Z bosons, as well as the photon). In addition, there are two 6-dimensional representations giving rise to new gauge particles called the X and the Y boson.