Sheet XII

Return by 19.12.2013

Question 1 [*Dimensions of* $\mathfrak{su}(3)$ *representations*]: The $\mathfrak{su}(3)$ representation with Dynkin label $[a, b]$ is associated with the Young diagram λ with two rows of length $r_1 = a + b$ and $r_2 = b$. The dimension of this representation equals the number of Young tableau fillings of λ with elements from the set $\{1, 2, 3\}$. Show that the dimension of the $\mathfrak{su}(3)$ representation $[a, b]$ equals

$$\dim\big([a, b]\big) = \frac{1}{2}(a + 1)(b + 1)(a + b + 2) \ . \tag{1}$$

Question 2 [*Construction of the simple algebra* $\mathfrak{g}_2$]: The exceptional Lie algebra $\mathfrak{g}_2$ is a 14-dimensional Lie algebra of rank 2. The purpose of this exercise is to construct it by extending the algebra $\mathfrak{sl}(3)$ in a suitable way; in the end you also need to check that the resulting commutators actually define a Lie algebra (i.e., that they satisfy the Jacobi identity).

(a) Start with the generators of $\mathfrak{sl}(3)$. Add to them 6 other generators that transform in the $[1, 0] \oplus [0, 1]$ representation of $\mathfrak{sl}(3)$; this set of 14 generators spans a basis for $\mathfrak{g}_2$. Draw the corresponding root diagram and check that there are 'short' roots and 'long' roots that have different lengths.

(b) Divide the roots of $\mathfrak{g}_2$ into positive and negative roots, and denote the positive roots by α_i with $i = 1, \ldots, 6$. Without loss of generality, we assume that the two simple roots (in terms of which all other positive roots are non-negative integer combinations) are α_1 and α_2. Moreover, the eigenvectors corresponding to the α_i are E_i, while those corresponding to $(-\alpha_i)$ are E_{-i}. Finally, we use as generators of the Cartan subalgebra H_1 and H_2, i.e., the Cartan generators associated with the two $\mathfrak{sl}(2)$ subalgebras corresponding to E_i and E_{-i} with $i = 1, 2$.
Define the generators corresponding to the other positive roots by iterating the adjoint action of the simple positive roots, and similarly for the negative roots.

(c) Identify all vanishing commutators.

(d) Check that the generators you have defined in point (b) satisfy the Jacobi identity.
Hint: Whenever possible, use that $[E_\alpha, E_\beta] = 0$ if $\alpha + \beta = 0$ is not a root.