# ② Lorentz & Poincaré symmetries in QFT

## 2.1. Lie groups

Lie group: group elements $g$ depend in continuous and differentiable way on a set of real parameters $\omega^a$, $a \in \{1, \dots N\}$

→ Choose $\omega^a$ s.t. $g(\omega^a = 0) = g(0) = e$

unit element of group

A (linear) representation $R$ of the group:

$$g \mapsto D_R(g)$$

abstract group element — linear operator

s.t. $D_R(e) = 1$

$$D_R(g_1) \cdot D_R(g_2) = D_R(g_1 \cdot g_2)$$

Typically: for finite dimensional representation, $D_R$ is $n \times n$ matrix

trsf of $\phi^i$: $\phi^i \to (D_R(g))^i{}_j \, \phi^j$

if $\phi^i$ transform under representation $R$

irreducible representation: no invariant subspace (all components mix)

For infinitesimal $\omega^a$, $D_R(g(\omega^a)) \simeq 1 + i\omega^a T^a_R$

$$T_R^a \equiv -i \left.\frac{\partial D_R(\omega^a)}{\partial \omega_a}\right|_{\omega=0}$$ : generators of group in representation $R$

Given $T_R^a$, for finite $\omega_a$: $D_R(g(\omega^a)) = e^{i\omega^a T^a_R}$

Lie algebra of group:

$$[T^a_{(R)}, T^b_{(R)}] = i f^{abc} T^c_{(R)}$$

structure constants

→ independent of representation $R$!

structure constants → Lie algebra → Lie group

To find matrix representations of group, find solutions to $[T^a, T^b] = i f^{abc} T^c$

<u>example</u>: rotations in $\mathbb{R}^3$

→ 3 parameters $\omega^a$: $\omega^1, \omega^2, \omega^3$

$$D_F(g(\omega_1=0, \omega_2=0, \omega_3)) = \begin{pmatrix} \cos\omega_3 & \sin\omega_3 & \\ -\sin\omega_3 & \cos\omega_3 & \\ & & 1 \end{pmatrix}$$

(fundamental rep. of rotation about $z(3)$ axis)

$$T_F^3 = -i \left.\frac{\partial D_F(\omega)}{\partial\omega_3}\right|_{\omega=0} = \begin{pmatrix} 0 & -i & \\ i & 0 & \\ & & 0 \end{pmatrix}$$

($= J_F^3$ notation)

$$[J^i, J^j] = i\epsilon^{ijk} J^k$$ Lie algebra satisfied by $J_F^3$

$D_F$: <u>fundamental rep.</u> → vectors $\vec{x}$ (under rotations)

<u>trivial representation</u>: $J_R^i$ ($= T_R^i$) $= 0$ (Lie algebra satisfied)

⇒ $D_R = \mathbb{1}$ → scalars (under rot.)

<u>spinor representation</u> $J_S^i = \frac{\sigma^i}{2}$ (Lie algebra satisfied)

→ spin ½

can classify quantities according to representation they transform under for rotations → generalize to Lorentz trsf.

## 2.2. Lorentz group

(homogeneous) LT : $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$

defined by requirement that $g_{\mu\nu} x^\mu x^\nu$ $(= t^2 - \vec{x}^2)$ invariant

$$\to \quad g_{\mu\nu} x'^\mu x'^\nu = \underbrace{g_{\mu\nu} \Lambda^\mu{}_\rho x^\rho \Lambda^\nu{}_\sigma x^\sigma}_{} \stackrel{!}{=} g_{\rho\sigma} x^\rho x^\sigma$$

$$g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\rho\sigma} \quad \text{or} \quad \Lambda^T \cdot g \cdot \Lambda = g$$

(i) $(\det \Lambda)^2 = 1$ $\quad$ $\det \Lambda = +1$ proper LT (connected to identity)

$\qquad\qquad\qquad\quad$ $\det \Lambda = -1$ improper LT (time reversal, parity)

(ii) $g_{\mu\nu} \Lambda^\mu{}_0 \Lambda^\nu{}_0 = 1 = (\Lambda^0{}_0)^2 - \sum_i (\Lambda^i{}_0)^2$

$\Lambda^0{}_0 \geq 1$ orthochronous LT

$\Lambda^0{}_0 \leq -1$ non-orthochronous LT

proper orthochronous LT are connected to identity

$$\Lambda^\mu{}_\nu = g^\mu{}_\nu + \omega^\mu{}_\nu \quad \text{with} \quad \omega^{\mu\nu} = -\omega^{\nu\mu} \quad (6 \text{ parameters})$$

→ form a subgroup (Lie group)

(often sloppy language: LT means prop. orth. LT)

any LT that is not proper & orthochronous can be written as a prop. orth. LT times $\mathcal{P}$, $\mathcal{T}$ or $\mathcal{P}\mathcal{T}$

$\mathcal{P}$ : parity $\quad P^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \quad (t, \vec{x}) \to (t, -\vec{x})$

$\mathcal{T}$ : time reversal $\quad T^\mu{}_\nu = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad (t, \vec{x}) \to (-t, \vec{x})$

$\mathcal{P}$ & $\mathcal{T}$ are <u>not</u> connected to identity

→ treated separately

generator of (proper, orthochronous) LT : $J^{\mu\nu}_R$

set of quantities $\phi^i$, $i=1 \ldots N$ transforms in representation $R$ of dim. $N$ under LT

$$\to \quad \phi^i \to \phi'^i = \underbrace{\left(e^{\frac{-i}{2}\omega_{\mu\nu} J^{\mu\nu}_R}\right)^i{}_j}\, \phi^j$$

convention: $\frac{1}{2}$ due to "double counting" in sum

$$= \phi^i + \underbrace{\frac{-i}{2}\omega_{\mu\nu}\left(J^{\mu\nu}_R\right)^i{}_j\, \phi^j}_{\Delta\phi^i}$$

consider vector representation $\phi^i \to V^\rho$

under LT: $V^\rho \to V'^\rho = \Lambda^\rho{}_\sigma V^\sigma = \left(g^\rho{}_\sigma + \omega^\rho{}_\sigma\right) V^\sigma$

$$\Rightarrow \Delta V^\sigma = \underbrace{\omega^\rho{}_\sigma V^\sigma} \overset{!}{=} \left(\frac{-i}{2}\right)\omega_{\mu\nu}\left(J^{\mu\nu}_V\right)^\rho{}_\sigma V^\rho$$

compare: $\left(J^{\mu\nu}_V\right)^\rho{}_\sigma = i\left(g^{\mu\rho} g^\nu{}_\sigma - g^{\nu\rho} g^\mu{}_\sigma\right)$

can use explicit form of $\left(J^{\mu\nu}_V\right)^\rho{}_\sigma$ to compute commutators

$\to$ <u>Lie algebra</u> of $SO(3,1)$

$$[J^{\mu\nu}, J^{\lambda\kappa}] = i\left(g^{\mu\kappa} J^{\nu\lambda} + g^{\nu\lambda} J^{\mu\kappa} - g^{\nu\kappa} J^{\mu\lambda} - g^{\mu\lambda} J^{\nu\kappa}\right)$$

not only for $J^{\mu\nu}_V$, but in all representations!

rearrange the 6 generators as follows:

$$\left.\begin{array}{l} J^i = \frac{1}{2}\varepsilon^{ijk} J^{jk} \\ \\ K^i \equiv J^{i0} \end{array}\right\} \qquad \begin{array}{ll} [J^i, J^j] = i\varepsilon^{ijk} J^k & \text{(ang. momentum)} \\ [J^i, K^j] = i\varepsilon^{ijk} K^k & \text{(K: spatial vector)} \\ [K^i, K^j] = -i\varepsilon^{ijk} J^k & \end{array}$$

Rewrite LT in terms of "new" generators:

$$\frac{1}{2}\omega_{\mu\nu} J^{\mu\nu} = \omega_{12} J^{12} + \omega_{13} J^{13} + \omega_{23} J^{23} + \sum_i \omega_{i0} J^{i0}$$

$$\equiv \vec{\theta}\cdot\vec{J} - \vec{\eta}\cdot\vec{K} = \theta^i J^i - \eta^i K^i$$

with $\theta^i \equiv \frac{1}{2}\varepsilon^{ijk}\omega^{jk}$ (i.e. $\theta^1 = \omega^{23} = \omega_{23}$ + cyclic)

angles of rotation

$\eta^i = \omega^{i0} = -\omega_{i0}$ boost parameters

$$\rightarrow \underset{R}{D}(\Lambda) = \underbrace{e^{-i\vec{\theta}\cdot\vec{J}_R + i\vec{\eta}\cdot\vec{K}_R}}_{\text{will be useful for spinor representations}} = e^{-\frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}_R}$$

## 2.3 Tensor representations

Construct representations from vector representation

$$(J^{\mu\nu}_{V})^{\rho}{}_{\sigma} = i\,(g^{\mu\rho} g^{\nu}{}_{\sigma} - g^{\nu\rho} g^{\mu}{}_{\sigma})$$

Of course there is the scalar representation $J^{\mu\nu}_{sc} = 0$ (trivial)

$\rightarrow\ D(\Lambda) = 1 \qquad s \rightarrow s' = D(\Lambda)\, s = s$

Vector rep. (as in 2.2) $\quad V \rightarrow V' = \underset{\uparrow\atop e^{-\frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}_{V}}}{D(\Lambda)} \cdot V \quad \leftrightarrow \quad V'^{\rho} = \Lambda^{\rho}{}_{\sigma} V^{\sigma}$

Generalize to arbitrary tensors, e.g. tensor of rank 2

$$T^{\mu\nu} \rightarrow T'^{\mu\nu} = \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} T^{\rho\sigma}$$

16 components $\rightarrow$ 16 dimensional representation, but reducible!

Split into antisymmetric $\quad A^{\mu\nu} = \frac{1}{2}(T^{\mu\nu} - T^{\nu\mu})$ $\quad$ (6 comp)

and symmetric $\quad S^{\mu\nu} = \frac{1}{2}(T^{\mu\nu} + T^{\nu\mu})$ $\quad$ part (10 comp)

these properties remain under LT!

in addition trace of $S^{\mu\nu}$ is L-inv

$$S = g_{\mu\nu} S^{\mu\nu} \rightarrow g_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} S^{\rho\sigma} = g_{\rho\sigma} S^{\rho\sigma} = S$$

def. of LT!

Tensor of rank 2 has rep.

$$4 \otimes 4 = 1 \oplus 6 \oplus 9$$

$4 \otimes 4$: tensor product of two 4-dim rep

$1$: 1-dim scalar

$6$: 6-dim anti-symmetric

↳ adjoint rep: same nr. as nr. of generators

$9$: 9-dim traceless (symmetric)

there are 2 <u>invariant tensors</u> (same coefficient in each inertial frame)

(i) $g^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ $\qquad g^{\mu\nu} \to g'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma g^{\rho\sigma} \xrightarrow{\text{def. of L-transf.}} = g^{\mu\nu}$

(ii) $\varepsilon^{\mu\nu\rho\sigma}$ totally antisymmetric tensor

$$\varepsilon^{\mu\nu\rho\sigma} \to \varepsilon'^{\mu\nu\rho\sigma} = \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} \Lambda^\rho{}_{\rho'} \Lambda^\sigma{}_{\sigma'} \varepsilon^{\mu'\nu'\rho'\sigma'} = \underbrace{(\det \Lambda)}_{= 1 \text{ for proper LT !}} \varepsilon^{\mu\nu\rho\sigma}$$

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We know how tensors transform under LT → deduce how they transform under rotations SO(3) → get information on ang. mom.

scalar: trivial trf. under SO(3) → ang. mom $j=0$

$V^\mu$: $V^0$ → trivial → $j=0$

$V^i$ → irreducible 3-dim rep of SO(3) → $j=1$

$V^\mu$ : $0 \oplus 1$

$T^{\mu\nu}$ : $(0 \oplus 1) \otimes (0 \oplus 1)$

$$= (0 \otimes 0) \oplus (0 \otimes 1) \oplus (1 \otimes 0) \oplus (1 \otimes 1)$$

$$= \underbrace{0}_{S} \oplus \underbrace{1 \oplus 1}_{A^{\mu\nu}} \oplus \underbrace{(0 \oplus 1 \oplus 2)}_{\text{traceless } S^{\mu\nu}}$$

(example $F^{\mu\nu} \Rightarrow \vec{E}, \vec{B}$)

in general tensors of rank N contains up to $j = N$

## 2.4. Spinor representations

recall subset of Lorentz algebra $[J^i, J^k] = i\varepsilon^{ijk} J^k$

→ representation by Pauli matrices $\frac{\sigma^i}{2} = J^i$

→ spin ½, | higher spin with tensor products such as
spinors $\psi_\alpha$ $\alpha \in \{1,2\}$ | $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$ etc.

how to generalize this to the relativistic case?
→ spinors in relativity

define $\vec{J}_\pm = \frac{1}{2}(\vec{J} \pm i\vec{K})$ (yet another combination to write the 6 generators)

Lie algebra becomes

$$\left.\begin{aligned} [J_+^i, J_+^j] &= i\varepsilon^{ijk} J_+^k \\ [J_-^i, J_-^j] &= i\varepsilon^{ijk} J_-^k \\ [J_+^i, J_-^j] &= 0 \end{aligned}\right\} \text{ 2 commuting copies of } SU(2)$$

- Representations can be labelled by $j_+$ and $j_-$ : $(j_-, j_+)$
- dimension of $(j_-, j_+)$ representation is $(2j_- + 1)(2j_+ + 1)$
- $\vec{J} = \vec{J}_+ + \vec{J}_-$, possible spin states in representation $(j_-, j_+)$ are $|j_+ - j_-|$ to $j_+ + j_-$ (addition of ang. mom.)

simplest case $(j_-, j_+) = (0,0)$

$$\left.\begin{aligned} \vec{J} &= \vec{J}_+ + \vec{J}_- = 0 \\ \vec{K} &= -i(\vec{J}_+ - \vec{J}_-) = 0 \end{aligned}\right\} \rightarrow \text{scalar representation}$$

$(j_-, j_+) = (0, \frac{1}{2})$ : right-handed Weyl spinor $\psi_R$

$\vec{J}^+ \to$ 2-dimensional $\Rightarrow \vec{J}^+ = \frac{\vec{\sigma}}{2}$

$\vec{J}^- \to$ 1 dimensional $\Rightarrow \vec{J}^- = 0$

$$\left.\begin{aligned} \vec{J} &= \vec{J}_+ + \vec{J}_- = \frac{\vec{\sigma}}{2} \\ \vec{K} &= -i(\vec{J}_+ - \vec{J}_-) = -i\frac{\vec{\sigma}}{2} \end{aligned}\right\} \quad D(\Lambda) = e^{-i\vec{\theta}\vec{J} + i\vec{\eta}\vec{K}} = e^{(-i\vec{\theta} + \vec{\eta})\frac{\vec{\sigma}}{2}}$$

An object that transforms under LT as

$$\psi_R \to \psi_R' = e^{(-i\vec{\theta} + \vec{\eta})\frac{\vec{\sigma}}{2}}\psi_R \equiv D_R(\Lambda)\psi_R$$

is called a right-handed Weyl spinor. It has 2 components

$(\psi_R)_\alpha \qquad \alpha \in \{1, 2\}$

(*) (In Literature, often $\dot{\alpha}$ used ; just notation, to indicate right-handed!)

$(j_-, j_+) = (\frac{1}{2}, 0)$ : left-handed Weyl spinor $\psi_L$

have $\vec{J}^+ = 0$ and $\vec{J}^- = \frac{\vec{\sigma}}{2}$

$$\Rightarrow \quad \vec{J} = \frac{\vec{\sigma}}{2}, \quad \vec{K} = +i\frac{\vec{\sigma}}{2} \quad \Rightarrow D(\Lambda) = e^{(-i\vec{\theta} - \vec{\eta})\frac{\vec{\sigma}}{2}}$$

An object that transforms under LT as

$$\psi_L \to \psi_L' = e^{(-i\vec{\theta} - \vec{\eta})\frac{\vec{\sigma}}{2}}\psi_L \equiv D_L(\Lambda)\psi_L$$

is called a left-handed Weyl spinor. 2 components $(\psi_L)_\alpha$

Note $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ are inequivalent representations

i.e. $\nexists$ matrix $S$, s.t. $D_R(\Lambda) = S D_L(\Lambda) S^{-1}$

rather $D_R(\Lambda) = S D_L^*(\Lambda) S^{-1}$ with $S = i\sigma_2$

(use $\sigma_2 \vec{\sigma}^* \sigma_2 = -\vec{\sigma}$ to show this) define charge conjugation (name will become clear later)

$\psi_L^c \equiv i\sigma_2 \psi_L^*$ claim this is a right-handed spinor

indeed: $\psi_L^c \to i\sigma^2 (D_L(\Lambda)\psi_L)^* = \underbrace{i\sigma^2 D_L^*(\Lambda)(i\sigma^2)^{-1}}_{D_R(\Lambda)} \underbrace{(i\sigma^2)\psi_L^*}_{\psi_L^c}$

Note: factor $i$ convention

similar: $\psi_R^c \equiv -i\sigma^2 \psi_R^*$ is a left-handed Weyl spinor

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Note: left-handed & right handed refers to transformation property, but is related to projection of spin along momentum (helicity) right-handed: $\vec{S} \uparrow\uparrow \vec{p}$ $\Big\}$ → later
$\vec{\sigma}\cdot\vec{p}$ left-handed $\vec{S} \uparrow\downarrow \vec{p}$

for $m=0$ helicity (→ chirality) is L inv!

Dirac spinor (→ section 3)

under parity: $\vec{K} \to -\vec{K}$ (vector) and $\vec{J} \to \vec{J}$ (axial vector)

→ $\vec{J}_+ \leftrightarrow \vec{J}_-$ representation $(j_-, j_+) \leftrightarrow (j_+, j_-)$

*not* basis for parity transformation.

if theory invariant under parity (e.g. QED) it is useful to work with fields which provide rep under (p&o) LT *and* parity

→ Dirac $\psi_\alpha = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ LT $\psi \to \begin{pmatrix} D_L(\Lambda) & 0 \\ 0 & D_R(\Lambda) \end{pmatrix} \psi$

$\alpha = 1 \ldots 4$ $\mathcal{P}$ $\psi \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \psi$ $\psi_L \leftrightarrow \psi_R$

if theory *not* invariant under parity (e.g. weak interactions) work with Weyl spinors

## 2.5 Field transformations

So far, we have considered transformations at a fixed point P

i.e. for $x \to x' = \Lambda x$ we have considered $\phi(x) \to \phi'(x') = D(\Lambda)\,\phi(x)$

scalar: $\phi'(x') = \mathbb{1} \cdot \phi(x)$

Vector $V'^{\mu}(x') = \Lambda^{\mu}{}_{\nu} V^{\nu}(x)$

(left-handed) Weyl spinor $(\psi_L')_\alpha (x') = \left(e^{(-i\vec{\theta} - \vec{\eta})\frac{\vec{\sigma}}{2}}\right)_\alpha{}^\beta (\psi_L)_\beta (x)$

These were all finite-dimensional representations (& linear)

⌈ Note: these rep. are not unitary, e.g. "missing" $i$ in the $\vec{\eta}\cdot\frac{\vec{\sigma}}{2}$ term. Related to fact that Lorentz group is not compact (rotations are, but boosts not) ⌋

For a field (e.g. a scalar field $\phi(x)$) we can also consider transformation at a fixed coordinate $x$

i.e. for $x \to x' = \Lambda x$ consider $\phi(x) \to \phi'(x)$
not $x'$

doing this, we compare the field $\phi$ at <u>different</u> points!
→ infinite-dimensional representations

$\delta\phi = \underbrace{\phi'(x) - \phi(x)}_{\text{fixed coordinate}}$ (compare $\Delta\phi = \underbrace{\phi'(x') - \phi(x)}_{\text{fixed point}}$)

<u>for scalar field:</u> $\phi'(x') = \phi(x) = \phi'(\Lambda x)$

$\Rightarrow \phi'(x) = \phi(\Lambda^{-1}x) \rightsquigarrow \phi(x - \delta x)$ (infinitesimal)

$\Rightarrow \delta\phi = \phi(x-\delta x) - \phi(x) = -\delta x^\mu \partial_\mu \phi(x)$

for a LT $\quad \delta x^\mu = \omega^\mu{}_\nu x^\nu$

$$\to \quad \delta\phi = -\omega^\mu{}_\nu x^\nu \partial_\mu \phi(x) = -\omega_{\mu\nu} x^\nu \partial^\mu \phi(x)$$

$$= -\frac{1}{2}\omega_{\mu\nu}\left(x^\nu\partial^\mu - x^\mu\partial^\nu\right)\phi(x) \overset{!}{=} -\frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}_{sc}\,\phi(x)$$

$$\to \quad J^{\mu\nu}_{sc} = \boxed{i\left(x^\mu\partial^\nu - x^\nu\partial^\mu\right) \equiv L^{\mu\nu}}$$

for a scalar field $\quad J^{\mu\nu}_{sc} = L^{\mu\nu}$ $\quad$ exercise: they satisfy Lie algebra

<u>for a vector field</u> $\quad V(x) \qquad V'(x) = D(\Lambda)\,V(\Lambda^{-1}x)$

$$\delta V = V'(x) - V(x) = V'(x') - V(x) + \left(V'(x) - V'(x')\right)$$

$$= \underbrace{\left(D(\Lambda) - 1\right)V(x)}_{\Delta V} + \underbrace{\left(V'(x) - V'(x) - \delta x^\mu \partial_\mu V'(x) + \dots\right)}_{\text{as for scalar field}}$$

$\Delta V$ $\quad = 0$ for scalar field
but $\neq 0$ in general!

as for scalar field $\to$ $-\frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\,V(x)$

$$D(\Lambda) = e^{-\frac{i}{2}\omega_{\mu\nu}\, \underbrace{i\left(g^{\mu\rho}g^\nu{}_\sigma - g^{\nu\rho}g^\mu{}_\sigma\right)}_{\equiv S^{\mu\nu}}} \to -\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}$$

before $J^{\mu\nu}_V$

$$\to \quad \delta V = -\frac{i}{2}\omega_{\mu\nu}\left(L^{\mu\nu} + S^{\mu\nu}\right) \equiv -\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}$$

$i(x^\mu\partial^\nu - x^\nu\partial^\mu)$, infinite-dim representation,

$i(g^{\mu\rho}g^\nu{}_\sigma - g^{\nu\rho}g^\mu{}_\sigma)$
4-dim representation
$\to$ specific to vector field!

| |
|---|
| Wigner: Inf.dim rep either unitary (linear) or antiunitary (antilinear) |

general:

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}$$

$S^{\mu\nu}$ $\leftarrow$ specific for spin (as in 2.3 and 2.4)

$L^{\mu\nu}$ $\uparrow$ independent on nature of field

## 2.6. The Poincaré group

Include inhomogeneous term in LT: $x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$

$\uparrow$ translation

2 successive Poinc trf: $x'' = \Lambda_2 x' + a_2 = \Lambda_2(\Lambda_1 x + a_1) + a_2$

$$D(\Lambda_2, a_2) \cdot D(\Lambda_1, a_1) = D(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2) \; ; \quad D^{-1}(\Lambda, a) = D(\Lambda^{-1}, -\Lambda^{-1}a)$$

apart from the 6 generators $J^{\mu\nu}$ for homogeneous LT we also have 4 generators $P^\mu$ for translations

Under translation: $\phi(x) \to \phi'(x) = \phi(x-a) = e^{ia_\mu P^\mu}\phi(x)$

scalar field $\phi'(x') = \phi'(x+a) = \phi(x)$

for infinitesimal $a_\mu$: $\delta\phi = -a_\mu \partial^\mu \phi = +i a_\mu P^\mu \phi$

$$\Rightarrow P^\mu = i\partial^\mu$$

Poincaré algebra

$[J^{\mu\nu}, J^{\kappa\lambda}] = \ldots$ as before

$[P^\mu, P^\nu] = 0$

$[P^\mu, J^{\kappa\lambda}] = i(g^{\mu\kappa}P^\lambda - g^{\mu\lambda}P^\kappa)$

This can be derived directly from

$D(1+\omega, \varepsilon) \equiv 1 + i\varepsilon^\mu P_\mu - \frac{i}{2}\omega^{\mu\nu}J_{\mu\nu}$ $\to$ exercise

for infinitesimal trf $\Lambda^\mu{}_\nu = g^\mu{}_\nu + \omega^\mu{}_\nu$ and $a^\mu = \varepsilon^\mu$

Here, $D$ is an infinite-dimensional unitary representation
$\to$ $P_\mu$ and $J_{\mu\nu}$ are hermitian

recall Wigner: antiunitary is also possible in general

Consider transformation of generators $P_\mu$ and $J_{\mu\nu}$

Since $D(1+\omega, \varepsilon)$ is an operator, it transforms like

$$D(\Lambda,a)\, D(1+\omega,\varepsilon)\, D^{-1}(\Lambda,a) = D(\Lambda,a)\, D(1+\omega,\varepsilon)\, D(\Lambda^{-1}, -\Lambda^{-1}a)$$

$$= D(\Lambda,a)\, D((1+\omega)\Lambda^{-1}, -(1+\omega)\Lambda^{-1}a + \varepsilon)$$

$$= D(\Lambda(1+\omega)\Lambda^{-1}, -\Lambda(1+\omega)\Lambda^{-1}a + \Lambda\varepsilon + a)$$

$$= D(\Lambda\omega\Lambda^{-1}, -\Lambda\omega\Lambda^{-1}a + \Lambda\varepsilon)$$

this means $D(\Lambda,a)\left(1 - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu} + i\varepsilon_\mu P^\mu\right) D^{-1}(\Lambda,a)$

$$= -\frac{i}{2}(\Lambda\omega\Lambda^{-1})_{\mu\nu} J^{\mu\nu} + i(\Lambda\varepsilon - \Lambda\omega\Lambda^{-1}a)_\mu P^\mu$$

compare terms $\sim \varepsilon_\mu$ :

$$D(\Lambda,a)\, i\varepsilon_\mu P^\mu D^{-1}(\Lambda,a) = i\Lambda_\mu{}^\rho \varepsilon_\rho P^\mu = i\Lambda_\nu{}^\mu \varepsilon_\mu P^\nu$$

$$\Rightarrow D(\Lambda,a)\, P^\mu D^{-1}(\Lambda,a) = \Lambda_\nu{}^\mu P^\nu = (\Lambda^{-1})^\mu{}_\nu P^\nu$$

compare terms $\sim \omega_{\mu\nu}$

$$D(\Lambda,a)\, \frac{-i}{2}\omega_{\mu\nu}J^{\mu\nu} D^{-1}(\Lambda,a) = \frac{-i}{2}\Lambda_\mu{}^\rho \omega_{\rho\sigma}(\Lambda^{-1})^\sigma{}_\nu J^{\mu\nu} - i\Lambda_\mu{}^\rho \omega_{\rho\sigma}(\Lambda^{-1})^\sigma{}_\nu a^\nu P^\mu$$

$$= -\frac{i}{2}\Lambda_\sigma{}^\mu \omega_{\mu\nu}\Lambda_\kappa{}^\nu J^{\sigma\kappa} - i\Lambda_\sigma{}^\mu \omega_{\mu\nu}\Lambda_\kappa{}^\nu a^\kappa P^\sigma$$

$$\Rightarrow D(\Lambda,a)\, J^{\mu\nu} D^{-1}(\Lambda,a) = \Lambda_\sigma{}^\mu \Lambda_\kappa{}^\nu \left(J^{\sigma\kappa} + a^\kappa P^\sigma - a^\sigma P^\kappa\right)$$

For $a^\mu = 0$ : $P^\mu$ is a vector
$J^{\mu\nu}$ is a tensor of rank 2

$P^\mu$ is translation invariant (not affected by $a^\mu$)

$J^{\mu\nu}$ is <u>not</u> translation invariant! (compare change of ang. momentum if origin is shifted)

Consider now parity $\mathcal{P}$ and time reversal $\mathcal{T}$

$\mathcal{P} \to D(P,0) \qquad \mathcal{T} \to D(T,0) \qquad P^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \quad T^\mu{}_\nu = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

$P^\rho$ transforms as follows:

$$\mathcal{P}\, iP^\rho\, \mathcal{P}^{-1} = iP^\rho{}_\nu\, P^\nu \quad \leftarrow \text{Notation } \mathcal{G}\ \ P^\rho{}_\nu \quad P^\nu \text{ mom op.}$$
$$\mathcal{T}\, iP^\rho\, \mathcal{T}^{-1} = iT^\rho{}_\nu\, P^\nu$$

want to sort out if $\mathcal{P}$ / $\mathcal{T}$ are unitary or antiunitary

Parity: $\quad \mathcal{P}\, iH \mathcal{P}^{-1} = P^0{}_\nu\, iP^\nu = iH \qquad (\text{take } \rho = 0)$

if $\mathcal{P}$ were antiunitary, then $-\mathcal{P} H \mathcal{P}^{-1} = H$ ??

$\to$ $\forall$ state with energy $E>0$ we would have a parity trsf. state with $E<0$ ↯

$\Rightarrow$ $\mathcal{P}$ is unitary $\qquad \mathcal{P} H \mathcal{P}^{-1} = H$

Time reversal $\qquad \mathcal{T}\, iH\, \mathcal{T}^{-1} = T^0{}_\nu\, iP^\nu = -iH$

now we need $\mathcal{T}$ antiunitary!

$$\mathcal{T}\, iH \mathcal{T}^{-1} = i^* \mathcal{T} H \mathcal{T}^{-1} = -i\, \mathcal{T} H \mathcal{T}^{-1} = -iH$$

$\hookrightarrow \quad \mathcal{T} H \mathcal{T}^{-1} = H$

Note: the theories we will consider in this course are invariant under $\mathcal{P}$ and $\mathcal{T}$, i.e. their Hamiltonian $H$ satisfies $\mathcal{P} H \mathcal{P}^{-1} = H$ and $\mathcal{T} H \mathcal{T}^{-1} = H$

but <u>not</u> all theories respect these symmetries (electroweak interactions e.g. don't!)

From $\mathcal{P}\, i P^\rho \mathcal{P}^{-1} = i \mathcal{P}^\rho{}_\nu P^\nu$, $\mathcal{T}\, i P^\rho \mathcal{T}^{-1} = i \mathcal{T}^\rho{}_\nu P^\nu$

$\mathcal{P}\, i J^{\rho\sigma} \mathcal{P}^{-1} = i \mathcal{P}^\rho{}_\mu \mathcal{P}^\sigma{}_\nu J^{\nu\mu}$, $\mathcal{T}\, i J^{\rho\sigma} \mathcal{T}^{-1} = i \mathcal{T}^\rho{}_\mu \mathcal{T}^\sigma{}_\nu J^{\mu\nu}$

We can deduce:

$$\mathcal{P} \vec{J} \mathcal{P}^{-1} = \vec{J} \quad \text{axial vector!}$$
$$\left.\begin{aligned} \mathcal{P} \vec{K} \mathcal{P}^{-1} &= -\vec{K} \\ \mathcal{P} \vec{P} \mathcal{P}^{-1} &= -\vec{P} \end{aligned}\right\} \text{vectors}$$
$$\mathcal{T} \vec{J} \mathcal{T}^{-1} = -\vec{J}$$
$$\mathcal{T} \vec{K} \mathcal{T}^{-1} = \vec{K}$$
$$\mathcal{T} \vec{P} \mathcal{T}^{-1} = -\vec{P}$$

consistent with Poincaré algebra!

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To label representation, find Casimir operators

(Casimir op: an operator that commutes with all generators)

claim: $\boxed{P_\mu P^\mu \text{ and } W_\mu W^\mu \text{ are Casimir op.}}$

where $W^\mu \equiv -\frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma$ Pauli-Lubanski vector

'proof': $P_\mu P^\mu$ and $W_\mu W^\mu$ are Linv ($J_{\nu\rho}$ ~ tensor of rank 2, $P_\sigma$ vector → covariance!) ⇒ they commute with $J^{\mu\nu}$

$\Rightarrow [P_\mu P^\mu, J_{\kappa\sigma}] = 0$ and $[W_\mu W^\mu, J_{\kappa\sigma}] = 0$

also $[P_\mu P^\mu, P^\kappa] = 0$ obvious and since

$[W^\mu, P^\kappa] = -\frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} [J_{\nu\rho}, P^\kappa] P_\sigma = -\frac{i}{2} \varepsilon^{\mu\nu\rho\sigma} (g_\rho{}^\kappa P_\nu - g_\nu{}^\kappa P_\rho) P_\sigma = 0$

We also have $[W_\mu W^\mu, P^\kappa] = 0$

Consider $P_\mu P^\mu = m^2$ with $m^2 > 0$ (massive particles)

what is $W_\mu W^\mu$ ? $\rightarrow$ go to rest frame $P^\mu = (m, \vec{0})$

$$W^\mu = -\frac{1}{2} m \, \varepsilon^{\mu\nu\rho 0} J_{\nu\rho} \quad \rightarrow \quad W^0 = 0$$

$$W^i = \frac{m}{2} \varepsilon^{0ijk} J^{jk} = m \cdot J^i$$

$$W_\mu W^\mu = -m^2 \vec{J}^2 \quad \rightarrow \quad -m^2 j(j+1)$$

for $m>0$, rep. characterized by $j$, $m_j = -j, \dots j$

states with $P^\mu = (m,0,0,0)$ left invariant by rotations $\rightarrow SU(2)$

(Little group: group of trsf that leaves $P^\mu$ invariant)

Consider $P_\mu P^\mu = 0$ (massless particles)

go to frame $P^\mu = (\omega, 0, 0, \omega)$

Little group: Euclidean group $E(2)$ (rotations $J_Z$ and translations in x & y direction)

naive (!!) limit $m \to 0$ from above:

$W_\mu W^\mu = 0$, in addition $P_\mu P^\mu = 0$ and $W_\mu P^\mu = 0$

$\Rightarrow$ $W^\mu \sim P$ $\rightarrow$ $W^\mu = h \cdot P^\mu$

actually $$W^\mu = \omega\,(J^3, \underbrace{J^1 - K^2, J^2 + K^1}, J^3) = \omega \cdot J_3 \, P^\mu = \frac{\vec{J}\cdot\vec{P}}{|\vec{P}|} P^\mu$$

correspond to a continuous d.o.f.

don't seem to be realized in nature

constrain to trivial $J^1 - K^2$ and $J^2 + K^1$

helicity, $h$

→ not all possible representations seem to play a rôle in nature !?!

There are further representations without known interpretations (some with continuous spin)

Summary of what we need:

Vacuum: $p^\mu = 0$, Little group $SO(3,1)$ (trivial)

massive particles $p^2 = m^2 > 0$, $p^0 > 0$

$\hookrightarrow$ $k^\mu = (m, 0, 0, 0)$ Little group $SO(3)$

additional characterization by spin → $(2s+1)$ states

massless particles $p^2 = 0$ $p^0 > 0$

$\hookrightarrow$ $k^\mu = (\omega, 0, 0, \pm\omega)$ Little group $E(2)$

additional characterization by helicity

$$h = \frac{\vec{p}\cdot\vec{J}}{|\vec{p}|} = \pm 1 \cdot |\vec{J}|$$

Note $h$ is pseudo scalar (changes sign under parity $\mathcal{P}$)

→ in parity conserving theories combine $h \pm |\vec{J}|$ states! 2 states